Spectral Statistics of Random d-regular Graphs

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Abstract

In this thesis we study the uniform random d-regular graphs on N vertices from a random matrix theory point of view.

In the first part of this thesis, we focus on uniform random d-regular graphs with large but fixed degree. In the bulk of the spectrum down to the optimal spectral scale, we prove that the Green's functions can be approximated by those of certain infinite tree-like (few cycles) graphs that depend only on the local structure of the original graphs. This result implies that the Kesten–McKay law holds for the empirical eigenvalue density down to the smallest scale and the bulk eigenvectors are completely delocalized. Our method is based on estimating the Green's function of the adjacency matrices and a resampling of the boundary edges of large balls in the graphs.

In the second part of this thesis, we prove, for $1 \ll d \ll N^{2/3}$, in the bulk of the spectrum the local eigenvalue correlation functions and the distribution of the gaps between consecutive eigenvalues coincide with those of the Gaussian orthogonal ensemble. In order to show this, we interpolate between the adjacent matrices of random d-regular graphs and the Gaussian orthogonal ensemble using a constrained version of Dyson Brownian motion.

CONTENTS

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1. Introduction

Random d-regular graphs are fundamental models of sparse random graphs and they arise naturally in many different contexts. The spectral properties of their adjacency matrices are of particular interest in computer science, combinatorics, and statistical physics. The relevant topics include the theory of expanders (see e.g. [80]), quantum chaos (see e.g. [82]), error-correcting codes (see e.g. [81, 84]) and graph ζ-functions (see e.g. [88]).

For a (uniform) random d-regular graph $\mathcal{G}$ on N vertices, we denote its adjacency matrix by $A = A(\mathcal{G})$. Thus A is uniformly chosen among all symmetric $N \times N$ matrices with entries in $\{0, 1\}$ with $\sum_j A_{ij} = d$ and $A_{ii} = 0$ for all i. Note that A has the *trivial* constant eigenvector with eigenvalue d. We also use the rescaled adjacency matrix $H = A/\sqrt{d-1}$, and we denote the set of (simple) d-regular graphs on N vertices by $\mathsf{G}_{N,d}$. In this thesis we study the spectral properties of uniform random d-regular graphs on N vertices, i.e. the eigenvalues and eigenvectors of H, from a random matrix theory point of view.

For random matrices of Wigner type, their spectral properties were extensively studied in the past twenty years. Precise estimates on the eigenvalues and eigenvectors of these matrices were well-understood (see e.g., [42, 86, 63, 36]):

(i) The empirical eigenvalue density is given by the semicircle law $\rho_{sc}(x) = \sqrt{[4 - x^2]_+}/(2\pi)$ on all scales larger than N^{-1}.

(ii) The normalized eigenvectors are uniformly bounded in ℓ^∞-norm by $N^{-1/2}$ (up to logarithmic correction).

(iii) The extremal eigenvalues are concentrated on scale $N^{-2/3}$.

(iv) Both bulk and edge universality holds; in particular, the distributions of the extremal eigenvalues are the same as those of Gaussian matrix ensembles (Tracy–Widom distributions).

The first three properties usually can be proved via estimates on the Green's function; the proofs of universality involve Dyson Brownian Motion or other comparison methods (see [36] for a review).

There are two major differences between Wigner type random matrices and the adjacency matrices of random d-regular graphs. Firstly, the entries of the adjacency matrices of random d-regular graphs are not independent. The entries have long range correlations, because of the restriction that the row sums and column sums are d. Secondly, for sparse random d-regular graphs, i.e. $d \ll N$, there is not much randomness. Because there are only d nonzero entries in each row and column in the adjacency matrices. However, extensive simulations indicate that (i)-(iv) hold for random regular graphs even with fixed degrees [75, 54, 76, 50].

For random d-regular graphs with *growing* degrees, i.e. $d \in [\xi^4, N^{2/3}\xi^{-2/3}]$, properties (i), (ii), were proved in [19]. One main result of this thesis is to show the bulk universality in (iv) holds for random d-regular graphs in the regime $d \in [N^{o(1)}, N^{2/3-o(1)}]$. In order to show this, we interpolate between the adjacency matrices of random d-regular graphs and the Gaussian orthogonal ensemble using a constrained version of Dyson Brownian motion.

For random d-regular graphs with *fixed* degree, previous results generally concern properties of eigenvalues and eigenvectors near the macroscopic scale. Weak versions of (i) and (ii) were proved in [33]: the empirical eigenvalue density is given by the Kesten–McKay law, $\frac{d}{d^2-x^2}\frac{1}{2\pi}\sqrt{[4(d-1)-x^2]_+}$ on the spectral scales $(\log N)^{-\Omega(1)}$; normalized eigenvectors are uniformly bounded in ℓ^∞-norm by $(\log N)^{-\Omega(1)}$. A weak version of (iii) that for any fixed $\varepsilon > 0$, the nontrivial eigenvalues of A are contained in $[-2\sqrt{d-1}-\varepsilon, 2\sqrt{d-1}+\varepsilon]$ was conjectured in [13] and proved in [46]; see also [21] for recent alternative arguments. It was also shown that the scale ε can actually be taken to be $\Omega((\log N)/\log\log N)$ in [21]. Another main accomplishment of this thesis is the proof that the Kesten–McKay law holds for the empirical eigenvalue density

down to the smallest scale and the bulk eigenvectors are completely delocalized, i.e. (i) and (ii) hold, provided d is larger than certain constant. For this, we introduce an approach that allows the tools developed from random matrix theory to make use of the local geometry of random regular graphs, while it also captures key random matrix behavior.

Notation. For two quantities X and Y depending on N, we use the notations $X = O_{\leqslant}(Y)$ if Y is positive and $|X| \leqslant Y$; $X = O(Y)$ if X, Y are positive and there exists some universal constant C such that $X \leqslant CY$; $X = o(Y)$, $X \ll Y$ or $Y \gg X$ if Y is positive and $\lim_{N \to \infty} X/Y = 0$; $X = \Omega(Y)$ if X, Y are positive and $\liminf_{N \to \infty} X/Y > 0$. We write $[\![a, b]\!] = [a, b] \cap \mathbb{Z}$ and $[\![N]\!] = [\![1, N]\!]$.

1.1. **Main results I: Spectral density and eigenvector delocalization.** In the

first part of this thesis, we consider random regular graphs of large but fixed degree d. We prove that the Kesten–McKay law holds for the empirical eigenvalue density down to the smallest scale and the bulk eigenvectors are completely delocalized.

It is well known that most regular graphs of a fixed degree $d \geqslant 3$ are *locally tree-like* in the sense that: (i) for any fixed radius R (and actually for $R = \Omega(\log_{d-1} N)$), the radius-R neighborhoods of almost all vertices are the same as those in the infinite d-regular tree; (ii) the R-neighborhoods of all vertices have bounded excess, which is the smallest number of edges that must be removed to yield a tree; see e.g. Proposition 2.1 below. The tree-like structure is important for the following results, valid in general for deterministic graphs and in some cases requiring randomness as well.

(i) For regular graphs with locally tree-like structure, the macroscopic spectral density of A converges to the Kesten–McKay law [59, 70], characterized by the density $\frac{d}{d^2 - x^2} \frac{1}{2\pi} \sqrt{[4(d-1) - x^2]_+}$. For random regular graphs, the Kesten–McKay law was established on spectral scales $(\log N)^{-\Omega(1)}$ [33, 48, 15] by using the fact that the locally tree-like structure holds with high probability in neighborhoods of radius $\Omega(\log_{d-1} N)$.

(ii) For regular graphs with locally tree-like structure, the eigenvectors v of A are weakly delocalized: their entries are uniformly bounded by $(\log N)^{-\Omega(1)}\|v\|_2$ [33, 48, 26] and their ℓ^2-mass cannot concentrate on a small set [26]. If, in addition, the graphs are expanders, the eigenvectors of A also satisfy the quantum ergodicity property [15, 25, 14].

(iii) For random regular graphs using the locally tree-like structure as important input, for any fixed $\varepsilon > 0$, the nontrivial eigenvalues of A are contained in $[-2\sqrt{d-1}-\varepsilon, 2\sqrt{d-1}+\varepsilon]$. This was conjectured in [13] and proved in [46]; see also [78, 21] for recent alternative arguments. It was also shown that the scale ε can actually be taken to be $\Omega((\log N)/\log\log N)$ in [21].

To take advantage of both the local tree-like structure and the random matrix-like structure, we use switchings to resample the boundaries of large balls (see Section 3.3). This operation preserves the local tree-like structure and it also captures sufficient global structure in random regular graphs. This resampling generalizes and adds a geometric component to the local resampling method introduced in [19] for random regular graphs with $d \gg \log N$. The idea of using some form of switchings in studying random regular graphs goes back at least to [71], where it was used in the enumeration of such graphs; see also [92] for further applications in enumeration. Finally, to analyze the propagation of the boundary effect to the interior of the ball in the Green's function, we explicitly compute the Green's function of the tree-like graphs.

1.1.1. *Spectral density.* With high probability, the spectral measure of the rescaled adjacency matrix $H = A/\sqrt{d-1}$ converges weakly to the rescaled Kesten–McKay law with density given by

$$(1.1) \qquad \rho_d(x) = \left(1 + \frac{1}{d-1} - \frac{x^2}{d}\right)^{-1} \frac{\sqrt{[4-x^2]_+}}{2\pi}.$$

This convergence can be expressed as $m(z) = m_d(z) + o(1)$ for any $z \in \mathbb{C}_+$ independent of N, where $m_d(z)$ is the Stieltjes transform of ρ_d, and $m(z)$ is the Stieltjes transform

of the empirical spectral measure of H,

$$(1.2) \qquad m_d(z) = \int \frac{1}{\lambda - z} \rho_d(\lambda) \, d\lambda, \qquad m(z) = \frac{1}{N} \sum_{j=1}^{N} \frac{1}{\lambda_j - z},$$

and $\mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Im}[z] > 0\}$ is the upper half-plane. The imaginary part of the spectral parameter $z \in \mathbb{C}_+$ determines the scale of the convergence. In particular, the convergence $m(z) \to m_d(z)$ for all fixed z corresponds to the convergence on the macroscopic scale, i.e., for intervals containing order N eigenvalues. The following theorem gives the convergence on the optimal mesoscopic scale $\operatorname{Im}[z] \gg 1/N$, away from the spectral edges at ± 2.

Theorem 1.1 (Local Kesten–McKay Law). *Fix $\alpha > 4$, $\omega \geqslant 8$ and $\sqrt{d-1} \geqslant (\omega + 1)2^{2\omega+45}$. Then with probability $1 - \mathrm{o}(N^{-\omega+8})$ with respect to the uniform measure on* $\mathsf{G}_{N,d}$,

$$(1.3) \qquad\qquad |m(z) - m_d(z)| = \mathrm{O}(\log N)^{-\alpha}$$

uniformly for

$$(1.4) \qquad z \in \mathcal{D} := \left\{ z \in \mathbb{C}_+ : \operatorname{Im}[z] \geqslant (\log N)^{48\alpha+1}/N, \quad |z \pm 2| \geqslant (\log N)^{-\alpha/2+1} \right\}.$$

While Theorem 1.1 shows that the spectral density (or its Stieltjes transform, which is the trace of the Green's function) does concentrate, the individual entries of the Green's function of the random regular graph with bounded degree do *not* concentrate; see also Remark 3.2 below. This is different from the typical examples in random matrix theory, and it is one of the reasons that the fixed degree graphs require a more delicate analysis. For example, the random regular graph contains a triangle with probability uniformly bounded from below. For graphs with bounded degree, triangles and other short cycles have a strong local influence on the elements of the Green's function, and thus the spectrum.
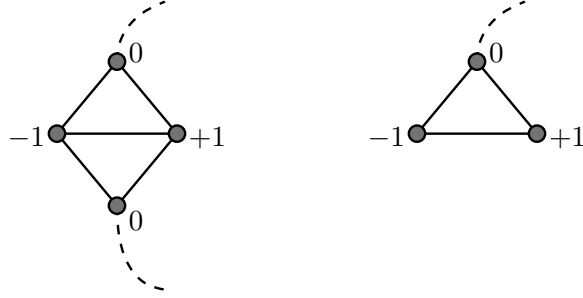
FIGURE 1. Theorem 1.2 shows that a random d-regular graph has only completely delocalized eigenvectors with probability $1 - \mathrm{o}(N^{-\omega+8})$. On the other hand, it is not difficult to show that a random d-regular graph has localized eigenvectors with probability $\Omega(N^{-d+2})$. For example, a random 3-regular graph contains the subgraph shown on the left with probability $\Omega(N^{-1})$. For comparison, also notice that an Erdős–Rényi graph with finite average degree contains localized eigenvectors with probability $\Omega(1)$; see the right figure.

1.1.2. *Eigenvectors.* The following theorem states that the eigenvectors in the bulk of the spectrum are completely delocalized.

Theorem 1.2 (Eigenvector delocalization)**.** *Fix $\alpha > 4$, $\omega \geqslant 8$ and $\sqrt{d-1} \geqslant (\omega + 1)2^{2\omega+45}$. Then, with probability $1 - \mathrm{o}(N^{-\omega+8})$ with respect to the uniform measure on $\mathsf{G}_{N,d}$, the eigenvectors v of H whose eigenvalue λ obeys $|\lambda \pm 2| \geqslant (\log N)^{1-\alpha/2}$ are simultaneously delocalized:*

$$
(1.5) \qquad \qquad \|v\|_\infty \leqslant \frac{\sqrt{2}(\log N)^{24\alpha+1/2}}{\sqrt{N}}\|v\|_2.
$$

Theorem 1.2 shows that with probability $1 - \mathrm{o}(N^{-\omega+8})$, the eigenvectors are completely delocalized. On the other hand, it is easy to see that, with probability $\Omega(N^{-d+2})$, the random d-regular graph has a localized eigenvector (see Figure 1). In particular, (1.5) cannot hold with probability higher than polynomial in $1/N$. Moreover, the Erdős–Rényi graph with finite average degree d has localized eigenvectors with probability $\Omega(1)$. Thus (1.5) with probability tending to 0 is false for the Erdős–Rényi graph with finite average degree d.

The delocalization of eigenvectors of (random and deterministic) regular graphs has been studied in [90, 33, 48, 15, 64, 26, 25, 14] (see also [77] for a survey of results on eigenvector delocalization in random matrices). Our result implies the optimal bound of order $1/\sqrt{N}$ (up to logarithmic corrections) on the ℓ^∞-norms of the (bulk) eigenvectors of random regular graphs.

For (deterministic) locally tree-like regular graphs, it was previously proved that the eigenvectors v are weakly delocalized in the sense that $\|v\|_\infty \leqslant (\log N)^{-c}\|v\|_2$ [33, 48, 26], and that eigenvectors cannot concentrate on a small set, in the sense that any vertex set $\mathbb{V} \subset [\![N]\!]$ with $\sum_{i \in \mathbb{V}} |v_i|^2 \geqslant \varepsilon\|v\|_2$ must have at least $N^{c(\varepsilon)}$ elements [26]. Moreover, for deterministic locally tree-like regular expander graphs, it was proved that the eigenvectors v satisfy a quantum ergodicity property: for all $a \in \mathbb{R}^N$ with $\|a\|_\infty \leqslant 1$ and $\sum_i a_i = 0$, averages of $|\sum_i a_i v_i^2|^2$ over many eigenvectors v are close to 0 [15, 25, 14].

1.2. **Main results II: Bulk Universality.** In the second part of this thesis, we focus on the regim $1 \ll d \ll N^{2/3}$. On this regime, it is known [90, 19, 33] that the eigenvalue density of $(d-1)^{-1/2}A$ converges to the Wigner *semicircle law* whose density is $\rho_{sc}(x) := \sqrt{[4-x^2]_+}/(2\pi)$. For d at least $(\log N)^4$, it was established in [19], the semi-circle law holds for the empirical eigenvalue density down to the smallest scale and the bulk eigenvectors are completely delocalized. Using the local semi-circle law as input, we prove that in the bulk of the spectrum the local eigenvalue correlation functions and the distribution of the gaps between consecutive eigenvalues coincide with those of the Gaussian orthogonal ensemble. In order to show this, we interpolate between the adjacency matrices of random d-regular graphs and the Gaussian orthogonal ensemble using a constrained version of Dyson Brownian motion.

As the adjacency matrix of a d-regular graph (RRG), the matrix A has the trivial uniform eigenvector $\mathbf{e} := N^{-1/2}(1,\ldots,1)^*$ with eigenvalue d. We denote by $\lambda_1 \geqslant \ldots \geqslant \lambda_{N-1}$ the ordered nontrivial eigenvalues of the rescaled adjacency matrix H,

and by $\mathbb{E}_{\mathrm{RRG}}$ the expectation with respect to the induced law on $\lambda_1 \geqslant \ldots \geqslant \lambda_{N-1}$. By comparison, we denote by $\mathbb{E}_{\mathrm{GOE}}$ the expectation with respect to the law of the ordered eigenvalues $\lambda_1 \geqslant \ldots \geqslant \lambda_{N-1}$ of the Gaussian Orthogonal Ensemble (GOE) on $\mathbb{R}^{(N-1)\times(N-1)}$, normalized so that the off-diagonal entries have variance N^{-1}.

The typical locations γ_i of the eigenvalues under the semicircle law are defined by

$$(1.6) \qquad \frac{i}{N} = \int_{\gamma_i}^2 \rho_{sc}(x)\,\mathrm{d}x\,.$$

Theorem 1.3. *Fix $\alpha > 0$, and suppose that $d \in [N^\alpha, N^{2/3-\alpha}]$. Then, in the limit $N \to \infty$, the bulk gap statistics of the random d-regular graph coincide with those of the GOE. More precisely, for any fixed $\kappa > 0$, $n \in \mathbb{N}$, and $\phi \in C_c^\infty(\mathbb{R}^n)$, we have*

$$(1.7) \qquad \left(\mathbb{E}_{RRG} - \mathbb{E}_{GOE}\right)\phi\left(N\rho_{sc}(\gamma_i)(\lambda_i - \lambda_{i+1}), \ldots, N\rho_{sc}(\gamma_i)(\lambda_i - \lambda_{i+n})\right) = \mathrm{o}(1)$$

as $N \to \infty$, uniformly in $i \in [\![\kappa N, (1-\kappa)N]\!]$.

Next, let $p_\#$ denote the symmetrized joint law of the eigenvalues of the ensemble $\# = RRG, GOE$. The correlation functions are defined for $n \in [\![1, N-1]\!]$ by

$$(1.8) \qquad p_\#^{(n)}(\mathrm{d}\lambda_1, \ldots, \mathrm{d}\lambda_n) := p_\#\left(\mathrm{d}\lambda_1, \ldots, \mathrm{d}\lambda_n, \mathbb{R}^{N-1-n}\right).$$

Theorem 1.4. *Fix $\alpha > 0$, and suppose that $d \in [N^\alpha, N^{2/3-\alpha}]$. Then, in the limit $N \to \infty$, the local correlation functions of the random d-regular graph coincide with those of the GOE. More precisely, for any fixed $n \in \mathbb{N}$, $\phi \in C_c^\infty(\mathbb{R}^n)$, and $E \in (-2, 2)$ we have*

(1.9)
$$\int_{\mathbb{R}^n} \phi(x_1, \ldots, x_n) N^n \left(p_{RRG}^{(n)} - p_{GOE}^{(n)}\right)\left(E + \frac{\mathrm{d}x_1}{N\rho_{sc}(E)}, \ldots, E + \frac{\mathrm{d}x_n}{N\rho_{sc}(E)}\right) = \mathrm{o}(1)\,.$$

For the GOE, the eigenvalue correlation functions are known explicitly; see e.g. [73]. Hence, the quantities for the GOE appearing on the left-hand sides of (1.7) and (1.9) can be computed explicitly. In fact, the eigenvalue gap distribution has

only been computed in the sense of averages over the gap index; for the GUE, the computation for a fixed gap was performed in [85].

The proofs of Theorems 1.3–1.4 follow the general three-step strategy developed in [42, 41, 40]; see e.g. [43] for a survey. In our setup, the strategy is formulated precisely in Section 4.1. The general idea is to study the convergence of eigenvalue statistics under Dyson Brownian motion (DBM) [34]. The three steps consist of (i) a *local law* providing precise estimates on the eigenvalue density down to the scale of individual eigenvalues, as well as the complete delocalization of the eigenvectors; (ii) the universality of the local eigenvalue statistics after the short time $t = N^{-1+\delta}$; and (iii) effective approximation of the local eigenvalue statistics of the original matrix ensemble at $t = 0$ by the one evolved up to time $t = N^{-1+\delta}$.

In all previous instances of the three-step strategy outlined above, the independence of the matrix entries was crucial for steps (i) and (iii). For the random regular graph, a new approach is required for both of these steps, the last one of which is the main content of this paper. The local law for random regular graphs was recently established in [19], thus performing step (i). As for step (ii), the convergence of the local eigenvalue statistics under DBM with deterministic initial data was recently established in [63], under the sole assumption that the eigenvalue density be bounded at the scale $N^{-1+\delta}$. Therefore the local semicircle law provides sufficient control on the eigenvalues so that using [63, 19] we can perform step (ii).

Thus, the main difficulty is step (iii). There are several known methods for performing this step, including Lindeberg's proof of the central limit theorem combined with higher moment matching conditions [86], or the Green's function comparison theorem [45]. For short times, a more direct method is to prove the stability of the eigenvalues under the DBM by analysing the dynamics of the individual matrix entries [24]. In all of these approaches, the independence of the matrix entries is used in an essential way. In contrast, the entries of random regular graphs are subject to

hard constraints, and are therefore not independent. Tracking carefully the dependence of the matrix entries (using the methods from [19]), we find that the eigenvalue evolution is stable under a *constrained DBM*, for times $t \leqslant N^{-1+\delta}$. Here, by *stability*, we mean that the changes in the local eigenvalue statistics are negligible.

This stability can also be interpreted as follows: there is a class of reasonably well-behaved observables, which completely characterize the local bulk eigenvalue statistics, and whose time evolution under the constrained DBM can be well approximated by a *switching dynamics* of random regular graphs. We note that it has been proposed that, for random regular graphs, the dynamics provided by DBM should be replaced with a switching dynamics; see in particular [58]. However, obtaining rigorous results on the local eigenvalue statistics using only a switching dynamics is difficult, because the induced eigenvalue process is neither continuous nor autonomous [57]. Our strategy crucially relies on the fact that the eigenvalue process under DBM is continuous and satisfies an autonomous system of SDEs.

Theorem 1.3 and Theorem 1.4 hold also for sparse random matrices with independent entries; see [52]. We will use parts of that analysis which are applicable here. The main effort and novelty of this paper is in the control of eigenvalues under constrained DBM up to time $t = N^{-1+\delta}$ using switchings.

1.3. **Related results.** Macroscopic eigenvalue statistics for random regular graphs of fixed degree have been studied using the techniques of Poisson approximation of short cycles [32, 56] and (non-rigorously) using the replica method [74]. These results show that the macroscopic eigenvalue statistics for random regular graphs of fixed degree are different from those of a Gaussian matrix. However, this is not predicted to be the case for the *local* eigenvalue statistics. Spectral properties of regular directed graphs have also been studied recently [28, 30].

The second largest eigenvalue λ_2 of regular graphs is of particular interest. For the case of fixed degree, see in particular [46, 78, 21, 47, 29]. The conjecture that the

distribution of the second largest eigenvalue on scale $N^{-2/3}$ is the same as that of the largest eigenvalue of the Gaussian Orthogonal Ensemble [80] would imply that slightly more than half of all regular graphs are Ramanujan graphs, namely d-regular graphs with $\lambda_2 \leqslant 2\sqrt{d-1}$ (for explicit and probabilistic constructions of sequences of Ramanujan graphs, see [66, 69, 68]). The spectrum of random regular graphs has also received interest from the study of ζ-functions, as it can be related by an exact relationship to the poles of the Ihara ζ-function of regular graphs [53, 18]; see also [88, 89].

Another interesting direction related to the spectral properties of random regular graphs concerns the phase diagram of the Anderson model. The model was originally defined on the square lattice $\mathbb{Z}^d$, but only limited progress was made for the delocalization problem in this setting. A simplified model on the infinite regular tree (Bethe lattice) is well-understood [60, 2, 4, 3, 10, 9, 8, 7, 6, 5]; see also [11] for a review. At large disorder, it is known that the Anderson model on the random regular graph exhibits Poisson statistics [49]. The eigenstates of the Anderson model on the random regular graph have also been studied in connection with many-body localization [31, 67].

In random matrix theory, the local spectral statistics of the generalized Wigner matrices are well understood; see in particular [55, 42, 41, 40, 45, 86, 44, 22, 38, 36]. Many results on local eigenvalue statistics also exist for Erdős-Rényi random graphs, in particular [39, 38, 52, 51]; the latter results apply down to logarithmically small average degrees. Similar results have also been proved for more general degree distributions [1, 12]. However, these types of results are false for the Erdős–Rényi graph with *bounded* average degree. For a review of other results for discrete random matrices, see also [91]. For the eigenvectors of random regular graphs with $d \in [N^{o(1)}, N^{2/3-o(1)}]$, the asymptotic normality was proved in [23]; see also the prior results for generalized Wigner matrices [61, 87, 24]. For random regular graphs of

fixed degree, a Gaussian wave correlation structure for the eigenvectors was predicted in [35] and partially confirmed in [17].

2. Geometry of Random d-regular graphs

2.1. **Graphs.** In this section we collect some definitions and terminologies about graphs, and basic structure properties of random d-regular graphs.

Graphs, adjacency matrices, Green's functions. Throughout this paper, graphs $\mathcal{G}$ are always simple (i.e., have no self-loops or multiple edges) and have vertex degrees at most d (non-regular graphs are also used). The geodesic distance (length of the shortest path between two vertices) in the graph $\mathcal{G}$ is denoted by $\text{dist}_{\mathcal{G}}(\cdot, \cdot)$. For any graph $\mathcal{G}$, the *adjacency matrix* is the (possibly infinite) symmetric matrix A indexed by the vertices of the graph, with $A_{ij} = A_{ji} = 1$ if there is an edge between i and j, and $A_{ij} = 0$ otherwise. Throughout the paper, we denote the *normalized adjacency matrix* by $H = A/\sqrt{d-1}$, where the normalization by $1/\sqrt{d-1}$ is chosen independently of the actual degrees of the graph. Moreover, we denote the (unnormalized) adjacency matrix of a directed edge (i,j) by e_{ij}, i.e. $(e_{ij})_{kl} = \delta_{ik}\delta_{jl}$. The *Green's function* of a graph $\mathcal{G}$ is the unique matrix $G = G(z)$ defined by $G(H-z) = I$ for $z \in \mathbb{C}_+$, where $\mathbb{C}_+$ is the upper half plane.

In Appendix B, several well-known properties of Green's function are summarized; they will be used throughout the paper. The Green's function $G(z)$ encodes all spectral information of H (and thus of A). In particular, the *spectral resolution* is given by $\eta = \text{Im}[z]$: the *macroscopic* behavior corresponds to η of order 1, the *mesoscopic* behavior to $1/N \ll \eta \ll 1$, and the *microscopic* behavior of individual eigenvalues corresponds to η below $1/N$.

Subsets and Subgraphs. Let $\mathcal{G}$ be a graph, and denote the set of its edges by the same symbol $\mathcal{G}$ and its vertices by $\mathbb{G}$. More generally, throughout the paper, we use blackboard bold letters for set or subsets of vertices, and calligraphic letters for

graphs or subgraphs. For any subset $\mathbb{X} \subset \mathbb{G}$, we define the graph $\mathcal{G}^{(\mathbb{X})}$ by removing the vertices $\mathbb{X}$ and edges adjacent to $\mathbb{X}$ from $\mathcal{G}$, i.e., the adjacency matrix of $\mathcal{G}^{(\mathbb{X})}$ is the restriction of that of $\mathcal{G}$ to $\mathbb{G} \setminus \mathbb{X}$. We write $G^{(\mathbb{X})}$ for the Green's function of $\mathcal{G}^{(\mathbb{X})}$. For any subgraph $\mathcal{X} \subset \mathcal{G}$, we denote by $\partial \mathcal{X} = \{v \in \mathbb{G} : \mathrm{dist}_{\mathcal{G}}(v, \mathcal{X}) = 1\}$ the *vertex boundary* of $\mathcal{X}$ in $\mathcal{G}$, and by $\partial_E \mathcal{X} = \{e \in \mathcal{G} : e \text{ is adjacent to } \mathcal{X} \text{ but } e \notin \mathcal{X}\}$ the *edge boundary* of $\mathcal{X}$ in $\mathcal{G}$. Moreover, for any subset $\mathbb{X} \subset \mathbb{G}$, we denote by $\partial \mathbb{X}$ and $\partial_E \mathbb{X}$ the vertex and edge boundaries of the subgraph induced by $\mathcal{G}$ on $\mathbb{X}$.

Neighborhoods. Given a subset $\mathbb{X}$ of the vertex set of a graph $\mathcal{G}$ and an integer $r > 0$, we denote the r-neighborhood of $\mathbb{X}$ in $\mathcal{G}$ by $\mathcal{B}_r(\mathbb{X}, \mathcal{G})$, i.e., it is the subgraph induced by $\mathcal{G}$ on the set $\mathbb{B}_r(\mathbb{X}, \mathcal{G}) = \{j \in \mathcal{G} : \mathrm{dist}_{\mathcal{G}}(\mathbb{X}, j) \leqslant r\}$. In particular $\mathcal{B}_r(i, \mathcal{G})$ is the radius-r neighborhood of the vertex i.

Moreover, given vertices i, j in $\mathcal{G}$ and $r > 0$, we denote by $\mathcal{E}_r(i, j, \mathcal{G})$ the smallest subgraph of $\mathcal{G}$ that contains all paths of length at most r between i and j. Namely,

(2.1) $\quad \mathcal{E}_r(i, j, \mathcal{G}) := \{e \in \mathcal{G} : \exists \text{ a path from } i \text{ to } j \text{ of length at most } r \text{ containing } e\}.$

Notice that $\mathcal{E}_{2r}(i, j, \mathcal{G}) \subset \mathcal{B}_r(i, \mathcal{G}) \cup \mathcal{B}_r(j, \mathcal{G})$.

Trees. The infinite d-regular tree is the unique (up to isometry) infinite connected d-regular graph without cycles, and is denoted by $\mathcal{Y}$. The rooted d-regular tree with root degree c is the unique (up to isometry) infinite connected graph that is d-regular at every vertex except for a distinguished root vertex o, which has degree c.

Kesten–McKay and semicircle law. Throughout this paper, the Stieltjes transforms of the Kesten–McKay law and that of the closely related semicircle law play an important role. Let $\rho_d(x)$ be the density of the (normalized) Kesten–McKay law (1.1) and $\rho_{sc}(x) := \frac{1}{2\pi} \sqrt{[4 - x^2]_+}$ that of Wigner's semicircle law. We denote their Stieltjes transforms by

(2.2) $$m_d(z) = \int \frac{\rho_d(x)}{x - z} \, dx, \quad m_{sc}(z) := \int \frac{\rho_{sc}(x)}{x - z} \, dx, \quad z \in \mathbb{C}_+.$$

Then $m_d(z)$ is explicitly related to $m_{sc}(z)$ by the equation (see also Proposition 2.6)

$$(2.3) \qquad m_d(z) = \frac{1}{-z - d(d-1)^{-1}m_{sc}(z)} = \frac{m_{sc}(z)}{1 - (d-1)^{-1}m_{sc}^2(z)}.$$

Moreover, it is well known that $m_{sc}(z)$ is a holomorphic bijection from the upper half plane $\mathbb{C}_+$ to the upper half unit disk $\mathbb{D}_+ := \{z \in \mathbb{C}_+ : |z| < 1\}$, and that it satisfies the algebraic equation

$$(2.4) \qquad z = -\left(m_{sc}(z) + \frac{1}{m_{sc}(z)}\right), \quad z \in \mathbb{C}_+,$$

and in particular that $|m_{sc}(z)| \leqslant 1$.

2.2. **Structure of random and deterministic regular graphs.** In this section, we collect some properties of random and deterministic regular graphs, which we use in the remainder of the paper.

Excess of random regular graphs. For any graph $\mathcal{G}$, we define its *excess* to be the smallest number of edges that must be removed to yield a graph with no cycles (a forest). It is given by

$$(2.5) \qquad \text{excess}(\mathcal{G}) := \#\text{edges}(\mathcal{G}) - \#\text{vertices}(\mathcal{G}) + \#\text{connected components}(\mathcal{G}).$$

There are different conventions for the normalization of the excess. Our normalization is such that the excess of a tree or forest is 0. Note that if $\mathcal{H} \subset \mathcal{G}$ is a subgraph, then $\text{excess}(\mathcal{H}) \leqslant \text{excess}(\mathcal{G})$. We will use the following well-known estimates for the excess in random regular graphs.

Proposition 2.1. *Let $\delta > 0$ and $\omega \geqslant 1$ be an integer. There is $\kappa > 0$ such that, if $R = \lfloor \kappa \log_{d-1} N \rfloor$, then the following holds for a uniformly chosen random d-regular graph $\mathcal{G}$ on $[\![N]\!]$, with probability at least $1 - o(N^{-\omega+\delta})$ for $N \geqslant N_0(d, \omega, \delta)$ large enough.*

- *All R-neighborhoods have excess at most ω:*

(2.6)   *for all $i \in [\![N]\!]$, the subgraph $\mathcal{B}_R(i, \mathcal{G})$ has excess at most ω.*

- *Most R-neighborhoods are trees:*

(2.7) $|\{i \in [\![N]\!] : \text{the subgraph } \mathcal{B}_R(i, \mathcal{G}) \text{ contains a cycle}\}| \leqslant N^\delta.$

In fact, one can take $\kappa < \delta/(2\omega + 2)$.

Proof. The statements are well known; for completeness, we sketch proofs in Appendix A.1. □

Excess and the number of non-backtracking walks. The next proposition bounds the number of non-backtracking walks (NBW) between two vertices in a graph in terms of the excess of the graph. Here a non-backtracking walk of length n is a sequence of vertices $(i_0, \ldots, i_n)$ such that the edge $\{i_{k-1}, i_k\}$ is adjacent to $\{i_k, i_{k+1}\}$ and such that the walks makes no steps backwards, i.e., $i_{k-1} \neq i_{k+1}$.

Proposition 2.2. *Let $\mathcal{G}$ be a graph with excess at most ω. Then the following hold.*

- *For any vertices $i, j \in \mathbb{G}$, and any $k \geqslant 1$, we have*

(2.8) $|\{NBW \text{ from } i \text{ to } j \text{ of length } \text{dist}_{\mathcal{G}}(i, j) + k - 1\}| \leqslant 2^{\omega k}.$

- *For any subgraph $\mathcal{H} \subset \mathcal{G}$ and two vertices i, j in $\mathcal{H}$ such that $\mathcal{E}_\ell(i, j, \mathcal{G}) \subset \mathcal{H}$, we have*

(2.9)
$|\{NBW \text{ from } i \text{ to } j \text{ of length } \ell + k \text{ which are not completely in } \mathcal{H}\}| \leqslant 2^{\omega(k+1)+1}.$

The graph $\mathcal{G}$ does not need to be regular or finite, and self-loops and multi-edges are allowed.
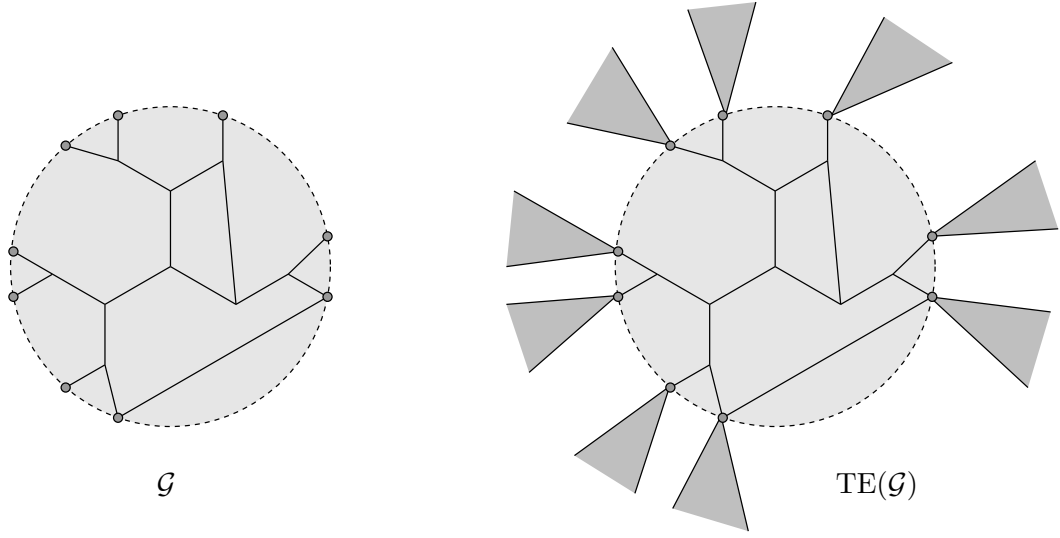
FIGURE 2. The left figure illustrates a finite graph $\mathcal{G}_0$; its extensible vertices are shown as grey circles. The right figure shows the tree extension $\mathrm{TE}(\mathcal{G}_0)$, in which a rooted tree (darkly shaded) is attached to every extensible vertex.

Proof. The statements are presumably also well known; lacking a reference, we include their proofs in Appendix A.2. $\qquad\square$

2.3. **Tree extension.** The local approximation of the Green's function of a graph will be defined in terms of the *tree extension*, defined next.

Definition 2.3 (deficit function)**.** *Given a graph $\mathcal{G}$ with vertex set $\mathbb{G}$ and degrees bounded by d, a* deficit function *for $\mathcal{G}$ is a function $g : \mathbb{G} \to [\![0, d]\!]$ satisfying $\deg_{\mathcal{G}}(v) \leqslant d - g(v)$ for all vertices $v \in \mathbb{G}$. We call a vertex $v \in \mathbb{G}$ extensible if $\deg_{\mathcal{G}}(v) < d - g(v)$.*

Definition 2.4 (tree extension)**.** *Let $\mathcal{G}_0$ be a finite graph with deficit function g.*

- *(i) The* tree extension *(abbreviated* TE*) of $\mathcal{G}_0$ is the (possibly infinite) graph $\mathrm{TE}(\mathcal{G}_0)$ defined by attaching to any extensible vertex v in $\mathcal{G}_0$ a rooted d-regular tree with root degree $d - g(v) - \deg_{\mathcal{G}_0}(v)$.*
- *(ii) The Green's function of $\mathcal{G}_0$ with tree extension, denoted $P(\mathcal{G}_0)$, is the Green's function of the (possibly infinite) graph $\mathrm{TE}(\mathcal{G}_0)$.*

See Figure 2 for an illustration of the tree extension. In our main result, we approximate the Green's function of a regular graph at vertices i, j by that of the tree extension of a neighbourhood of i, j. This requires specification of a deficit function, which we will usually do using the following conventions for deficit functions, assumed throughout the paper.

Conventions for deficit functions. Throughout this paper, all graphs $\mathcal{G}$ are equipped with a deficit function g. The interpretation of the deficit function $g(v)$ is that it measures the difference to the desired degree of the vertex v. We use the following conventions for deficit functions.

- If the deficit function of $\mathcal{G}$ is not specified explicitly, it is given by $g(v) = d - \deg_{\mathcal{G}}(v)$. Thus no vertex is extensible and the tree extension of $\mathcal{G}$ is trivial: $\mathcal{G} = \mathrm{TE}(\mathcal{G})$.

- If $\mathbb{X}$ is a subset of the vertices of $\mathcal{G}$, and g is the deficit function of $\mathcal{G}$, then the deficit function g' of $\mathcal{G}^{(\mathbb{X})}$ is given by $g'(v) = g + \deg_{\mathcal{G}}(v) - \deg_{\mathcal{G}^{(\mathbb{X})}}(v)$, unless specified explicitly. Thus when removing the edges incident to $\mathbb{X}$ from $\mathcal{G}$, these are also absent in the tree extension.

- If $\mathcal{H} \subset \mathcal{G}$ is a subgraph (which was not obtained as $\mathcal{G}^{(\mathbb{X})}$), then the deficit function of $\mathcal{H}$ is given by the restriction of the deficit function of $\mathcal{G}$ on $\mathcal{H}$, unless specified explicitly. Thus any vertex v in $\mathcal{H} \subset \mathcal{G}$ has the same degree in the tree extension $\mathrm{TE}(\mathcal{H})$ as in $\mathrm{TE}(\mathcal{G})$.

The above conventions are illustrated in Figure 3. In particular, in the case that $\mathcal{G}$ is a d-regular graph, the deficit function is always $g \equiv 0$, so that $\mathrm{TE}(\mathcal{G}) = \mathcal{G}$. Moreover, by our conventions, the tree extension of a subgraph $\mathcal{H} \subset \mathcal{G}$ is again a d-regular graph.

Definition 2.5. *Given an integer $r > 0$, we call $P_{ij}(\mathcal{E}_r(i, j, \mathcal{G}))$ the* localized Green's function *of $\mathcal{G}$ at vertices i, j.*
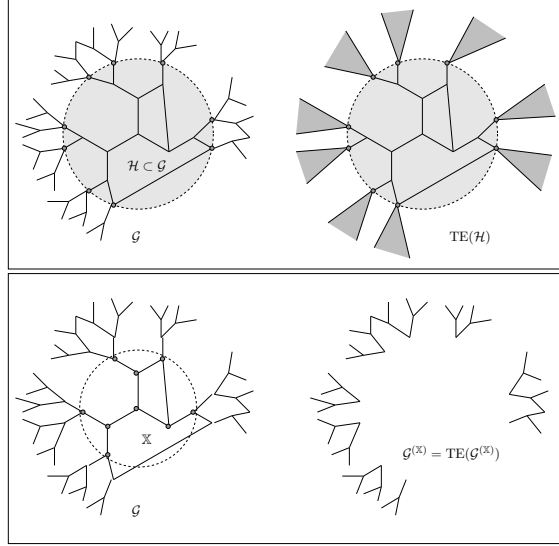
FIGURE 3. Given a graph $\mathcal{G}$ (with the standard deficit function $g = d - \deg_{\mathcal{G}}$), the left figure illustrates a subgraph $\mathcal{H} \subset \mathcal{G}$, which by our conventions inherits its deficit function from $\mathcal{G}$ by restriction. Thus all vertices in $\mathcal{H}$ have the same degrees in the tree extension $\mathrm{TE}(\mathcal{H})$ as in $\mathcal{G} = \mathrm{TE}(\mathcal{G})$. The right figure illustrates the graph $\mathcal{G}^{(\mathbb{X})}$ obtained by removing a vertex set $\mathbb{X}$. By our convention on the deficit function, the tree extension of $\mathcal{G}^{(\mathbb{X})}$ is then trivial.

For the infinite regular tree and for the rooted infinite regular tree with given root degree, it is elementary to compute the Green's function explicitly, as done in the following proposition.

Proposition 2.6. *Let $\mathcal{Y}$ be the infinite d-regular tree. For all $z \in \mathbb{C}_+$, its Green's function is*

$$(2.10) \qquad G_{xy}(z) = m_d(z) \left(-\frac{m_{sc}(z)}{\sqrt{d-1}} \right)^{\mathrm{dist}_{\mathcal{Y}}(x,y)} .$$

Let $\mathcal{Y}_0$ be the rooted infinite d-regular tree with root degree $d - 1$. Its Green's function is

$$(2.11) \qquad G_{xy}(z) = m_d(z) \left(1 - \left(-\frac{m_{sc}(z)}{\sqrt{d-1}} \right)^{2\ell(x,y)+2} \right) \left(-\frac{m_{sc}(z)}{\sqrt{d-1}} \right)^{\mathrm{dist}_{\mathcal{Y}_0}(x,y)} ,$$

where $\ell(x,y)$ is the depth of the common ancestor of the vertices x and y in $\mathcal{Y}_0$. In particular, if x is the root of $\mathcal{Y}_0$, then $G_{xx}(z) = m_{sc}(z)$.

The proof is given below. More general results for Green's functions on regular trees are discussed e.g. in [9, Section 3] and references given there.

Proposition 2.7. *Let $\omega \geqslant 6$ and $\sqrt{d-1} \geqslant 2^{\omega+2}$. Let $\mathcal{G}_0$ be a finite graph with vertex set $\mathbb{G}_0$ and deficit function g. Assume that (i) any connected component of $\mathcal{G}_0$ has excess at most ω, and that (ii) the sum of deficit function over any connected component of $\mathcal{G}_0$ satisfies $\sum g(v) \leqslant 8\omega$. Then the following holds for all $z \in \mathbb{C}_+$ and all $i,j \in \mathbb{G}_0$.*

(i) The Green's function $P_{ij}(\mathcal{G}_0)$ of $\mathrm{TE}(\mathcal{G}_0)$ satisfies

$$(2.12) \qquad |P_{ij}(\mathcal{G}_0, z)| \leqslant 2^{\omega+2} |m_{sc}(z)| q^{\mathrm{dist}_{\mathcal{G}_0}(i,j)},$$

and the diagonal terms satisfy the better estimate

$$(2.13) \qquad |P_{ii}(\mathcal{G}_0, z) - m_d(z)| \leqslant \frac{|m_{sc}(z)|}{4}.$$

(ii) Let $\mathcal{H}_0 \subset \mathcal{G}_0$ be a subgraph with vertex set $\mathbb{H}_0$. Then for any two vertices i,j in $\mathcal{H}_0$ such that $\mathcal{E}_\ell(i,j) \subset \mathcal{H}_0$, the ij-th entries of the Green's functions of the tree extensions of $\mathcal{G}_0$ and $\mathcal{H}_0$ satisfy

$$(2.14) \qquad |P_{ij}(\mathcal{G}_0, z) - P_{ij}(\mathcal{H}_0, z)| \leqslant 2^{2\omega+3} |m_{sc}(z)| q^{\ell+1}.$$

Item (i) states that $P_{ij}(\mathcal{G}_0)$ is bounded and has (up to constants) the same decay as the Green's function of the infinite d-regular tree $\mathcal{Y}$. In particular, (2.12) and (2.13) together with (2.3) imply that

$$(2.15) \qquad |P_{ij}(\mathcal{G}_0, z)| \leqslant (1 + \delta_{ij}/2) |m_{sc}(z)|.$$

Item (ii) states that $P_{ij}(\mathcal{G}_0)$ depends only weakly on $\mathcal{G}_0$. Especially, it implies the the following principle, which is used repeatedly throughout Sections 3.5–3.10.

Remark 2.8 (Localization principle)**.** *Let $\mathbb{X}$ be a (small) set of vertices in a graph $\mathcal{G}$. For vertices $i, j \in \mathbb{X}$, it is often convenient to replace $P_{ij}(\mathcal{E}_r(i, j, \mathcal{G}))$, namely the ij-th entry of the Green's function of the graph $\mathrm{TE}(\mathcal{E}_r(i, j, \mathcal{G}))$ which itself depends on i, j, by $P_{ij}(\mathcal{G}_0)$ of a graph $\mathcal{G}_0$ which is independent of i, j and contains $\mathcal{E}_r(i, j, \mathcal{G})$ for $i, j \in \mathbb{X}$. In this situation, we abbreviate $P = P(\mathcal{G}_0)$. The estimate (2.14) then implies that P_{ij} and $P_{ij}(\mathcal{E}_r(i, j, \mathcal{G}))$ are close in the sense*

$$(2.16) \qquad |P_{ij}(\mathcal{E}_r(i, j, \mathcal{G})) - P_{ij}| \leqslant 2^{2\omega+3}|m_{sc}|q^{r+1}$$

provided the assumptions of (2.14) are obeyed.

2.3.1. *Proof of Proposition 2.6.* The proof of Proposition 2.6 is a straightforward consequence of the Schur complement formula (B.4).

Proof. Let $\operatorname{dist}_{\mathcal{Y}}(x, y) = 1$. The Schur complement formula implies

$$(2.17) \qquad G_{yy}^{(x)} = \frac{-1}{z + (d-1)^{-1}\sum_{k,l \in \partial y \setminus \{x\}} G_{kl}^{(xy)}} = \frac{-1}{z + (d-1)^{-1}\sum_{k \in \partial y \setminus \{x\}} G_{kk}^{(y)}},$$

where ∂y is the set of adjacent vertices of y in $\mathcal{Y}$. By homogeneity, $G_{yy}^{(x)}$ is independent of x and y if $\operatorname{dist}_{\mathcal{Y}}(x, y) = 1$ and therefore equal to the unique solution to the equation $m = -1/(z + m)$ with $\operatorname{Im} m > 0$, which is m_{sc}. Applying the Schur complement formula again, it follows that

$$(2.18) \qquad G_{xx} = \frac{-1}{z + d(d-1)^{-1}m_{sc}} = m_d.$$

This proves (2.10) and (2.11) for $x = y$. The case $\text{dist}_\mathcal{Y}(x, y) = 1$ then follows, e.g., from

$$(2.19) \qquad 1 = \sum_y G_{xy}(H_{yx} - z\delta_{yx}) = \frac{d}{\sqrt{d-1}}G_{xy} - zm_d,$$

which, using $1 + zm_d + \frac{d}{d-1}m_d m_{sc} = 0$, implies

$$(2.20) \qquad G_{xy} = \frac{\sqrt{d-1}}{d}(1 + zm_d) = -\frac{m_d m_{sc}}{\sqrt{d-1}},$$

as claimed. The general case is similar by induction. $\qquad\square$

2.3.2. *Proof of Proposition 2.7 for $g \equiv 0$.* For the proof of Proposition 2.6, we require the notion of covering of a graph. Given a graph $\mathcal{G}$, a graph $\tilde{\mathcal{G}}$ together with a surjective map $\pi : \tilde{\mathbb{G}} \to \mathbb{G}$ is a *covering* of $\mathcal{G}$ if for each $x \in \tilde{\mathbb{G}}$, the restriction of π to the neighborhood of x is a bijection onto the neighborhood of $\pi(x)$ on $\mathbb{G}$. Every d-regular graph is covered by the infinite d-regular tree $\mathcal{Y}$ which is its universal covering.

The Green's functions of a graph $\mathcal{G}$ and a cover $\tilde{\mathcal{G}}$ with covering map $\pi : \tilde{\mathbb{G}} \to \mathbb{G}$ obey the following identity. For each $x \in \tilde{\mathbb{G}}$ and $\pi(x) = i \in \mathbb{G}$, we have

$$(2.21) \qquad G_{ij}(z) = \sum_{y:\pi(y)=j} \tilde{G}_{xy}(z),$$

if the right-hand side is summable (see Appendix B for the elementary proof of (2.21)). In particular, if $\mathcal{G}$ is an infinite simple d-regular graph and $\pi : \mathbb{Y} \to \mathbb{G}$ its universal covering map, where $\mathbb{Y}$ are the vertices of $\mathcal{Y}$, then by (2.10) and (2.21), for any vertex $x \in \mathbb{Y}$ such that $\pi(x) = i$, the resolvent entries of the graph $\mathcal{G}$ are given

by

$$G_{ij} = m_d \sum_{y:\pi(y)=j} \left(-\frac{m_{sc}}{\sqrt{d-1}} \right)^{\mathrm{dist}_{\mathcal{Y}}(x,y)}$$

(2.22)
$$= m_d \sum_{k \geq \mathrm{dist}_{\mathcal{G}}(i,j)} |\{\text{non-backtracking paths from } i \text{ to } j \text{ of length } k\}| \left(-\frac{m_{sc}}{\sqrt{d-1}} \right)^k.$$

For the number of non-backtracking paths, recall the estimates of Proposition 2.2. Using these, the proofs of (2.12) and (2.14) are straightforward from (2.22) if $g \equiv 0$.

Proof of (2.12) *for* $g \equiv 0$. For vertices i, j in different connected components of $\mathcal{G}_0$, we have $P_{ij}(\mathcal{G}_0) = 0$ and there is nothing to prove. Therefore, we can assume that i and j are in the same connected component.

Since we assume $g \equiv 0$, the tree extension $\mathcal{G}_1 = \mathrm{TE}(\mathcal{G}_0)$ is d-regular, and (2.22) implies

(2.23)
$$P_{ij}(\mathcal{G}_0, z) = m_d \sum_{k \geq \mathrm{dist}(i,j)} |\{\text{NBW in } \mathcal{G}_1 \text{ from } i \text{ to } j \text{ of length } k\}| \left(-\frac{m_{sc}}{\sqrt{d-1}} \right)^k.$$

Since $\mathcal{G}_0$ has excess at most ω, the same is true for $\mathcal{G}_1$. By the estimates for the number of non-backtracking paths from Proposition 2.2, the right-hand side of (2.22) is summable, provided that $\sqrt{d-1} \geq 2^{\omega+2}$, and

$$|G_{ij}| \leq |m_d| \sum_{k \geq 1} 2^{\omega k} q^{\mathrm{dist}_{\mathcal{G}_0}(i,j)+k-1} = |m_d| 2^\omega q^{\mathrm{dist}_{\mathcal{G}_0}(i,j)} \sum_{k \geq 1} (2^\omega q)^{k-1}$$

$$\leq |m_d| 2^\omega q^{\mathrm{dist}_{\mathcal{G}_0}(i,j)} \sum_{k \geq 0} 4^{-k} \leq 2^{\omega+1} |m_{sc}| q^{\mathrm{dist}_{\mathcal{G}_0}(i,j)}.$$

This completes the proof if $g \equiv 0$. $\square$

Proof of (2.14) *for* $g \equiv 0$. As in the proof of (2.12), we can assume that i and j are in the same connected component of $\mathcal{G}_0$. By (2.22), since all the non-backtracking

paths from i to j of length $\leqslant \ell$ are contained in $\mathcal{H}_0$, we have

$$P_{ij}(\mathcal{G}_0, z) - P_{ij}(\mathcal{H}_0, z)$$

$$= m_d \sum_{k \geqslant 1} |\{\text{NBW from } i \text{ to } j \text{ of length } \ell + k, \text{ not completely in } \mathcal{H}_0\}| \left(-\frac{m_{sc}}{\sqrt{d-1}} \right)^{\ell+k}.$$

By (2.9), we therefore have

$$|P_{ij}(\mathcal{G}_0, z) - P_{ij}(\mathcal{H}_0, z)| \leqslant |m_d| \sum_{k=1}^{\infty} 2^{\omega(k+1)+1} q^{\ell+k}$$

$$= |m_d| 2^{2\omega+1} q^{\ell+1} \sum_{k=1}^{\infty} (2^\omega q)^{k-1} \leqslant 2^{2\omega+2} |m_{sc}| q^{\ell+1},$$

again provided that $\sqrt{d-1} \geqslant 2^{\omega+2}$. This completes the proof if $g \equiv 0$. $\qquad \square$

2.3.3. *Proof of Proposition 2.7 for $g \not\equiv 0$.* To extend the bounds (2.12) and (2.14) to $g \not\equiv 0$, we use an alternative representation of $P_{ij}(\mathcal{G}_0)$ given as follows. In Definition 2.4, $P_{ij}(\mathcal{G}_0, z)$ is defined as the Green's function of the infinite graph obtained by attaching a d-regular tree at every extensible vertex of $\mathcal{G}_0$. The next lemma shows that it is equivalently given by attaching to every extensible vertex a self-loop with z-dependent complex weight. The proof of the lemma follows by application of the Schur complement formula.

Lemma 2.9. *Let $z \in \mathbb{C}_+$. Then for vertices $i, j \in \mathbb{G}_0$,*

$$P_{ij}(\mathcal{G}_0, z) = (H_2 - z)^{-1}$$

where H_2 is the normalized z-dependent adjacency matrix obtained by attaching to any extensible vertex v in $\mathcal{G}_0$ a self-loop with complex weight $-m_{sc}(z)(d - g(v) - \deg_{\mathcal{G}_0}(v))/\sqrt{d-1}$.

Proof. Let $\mathcal{G}_1 = \mathrm{TE}(\mathcal{G}_0)$, and denote the normalized adjacency matrix of $\mathcal{G}_0$ and $\mathcal{G}_1$ by H_0 and H_1 respectively. Then H_1 has the block form

$$
H_1 = \begin{bmatrix} H_0 & B' \\ B & D \end{bmatrix}
$$

where D is the normalized adjacency matrix of several copies of $\mathcal{Y}_0$, i.e. infinite d-regular tree with root degree $d-1$, and B_{xy} is $1/\sqrt{d-1}$ if y is an extensible vertex of $\mathcal{G}_0$ and x the root of one of the former copies of the tree $\mathcal{Y}_0$, and $B_{xy} = 0$ otherwise. By the Schur complement formula (B.3), it follows that, for any $i, j \in \mathbb{G}_0$,

$$
G_{ij}(\mathcal{G}_1, z) = (H_1 - z)_{ij}^{-1} = (H_0 - z - B'(D - z)^{-1}B)_{ij}^{-1}.
$$

Since $B'(D - z)^{-1}B$ is a diagonal matrix, indexed by the extensible vertices in $\mathcal{G}_0$ (which are disjoint), and since B is normalized by $1/\sqrt{d-1}$, it follows from (2.11) with $x = y$ that

$$
(B'(D - z)^{-1}B)_{vv} = m_{sc}\mathbf{1}_{v \text{ is extensible}}\frac{d - g(v) - \deg_{\mathcal{G}_0}(v)}{d - 1}.
$$

Thus $H_2 = H_0 - B'(D - z)^{-1}B$ and the claim of the lemma follows. $\square$

As previously, we abbreviate $\mathcal{G}_1 = \mathrm{TE}(\mathcal{G}_0)$, and denote by $\mathcal{G}_2$ the finite z-dependent graph with complex weight obtained by attaching at each extensible vertex v of $\mathcal{G}_0$ a self-loop with weight $-m_{sc}(z)(d - g(v) - \deg_{\mathcal{G}_0}(v))/\sqrt{d-1}$. Moreover, to extend (2.12) and (2.14) from $g \equiv 0$ to $g \not\equiv 0$, we denote by $\mathcal{G}_0'$ the same graph as $\mathcal{G}_0$ but with deficit function $g \equiv 0$, by $\mathcal{G}_1' = \mathrm{TE}(\mathcal{G}_0')$ its tree extension, and by $\mathcal{G}_2'$ the finite z-dependent graph with complex weight obtained by attaching at each extensible vertex v of $\mathcal{G}_0'$ a self-loop with weight $-m_{sc}(z)(d - \deg_{\mathcal{G}_0}(v))/\sqrt{d-1}$. We denote the normalized adjacency matrices of $\mathcal{G}_2$ and $\mathcal{G}_2'$ by H_2 and H_2' respectively.

Proof of (2.12) *for* $g \not\equiv 0$. By Lemma 2.9 and the case $g \equiv 0$, we have

$$\Gamma' := \max_{i,j \in \mathbb{G}_0} |G_{ij}(\mathcal{G}'_2, z)| \left(|m_{sc}| q^{\mathrm{dist}_{\mathcal{G}_0}(i,j)} \right)^{-1} \leqslant 2^{\omega+1}.$$

Our goal is to estimate

$$\Gamma := \max_{i,j \in \mathbb{G}_0} |G_{ij}(\mathcal{G}_2, z)| \left(|m_{sc}| q^{\mathrm{dist}_{\mathcal{G}_0}(i,j)} \right)^{-1}.$$

Notice that $H_2 - H'_2$ is a diagonal matrix with entries

$$(2.24) \qquad\qquad (H_2 - H'_2)_{vv} = \frac{m_{sc} g(v)}{d-1}, \quad v \in \mathbb{G}_0,$$

and the resolvent formula (B.1) implies

$$(2.25) \qquad G(\mathcal{G}'_2, z)_{ij} - G(\mathcal{G}_2, z)_{ij} = \sum_{v \in \mathbb{G}_0} G_{iv}(\mathcal{G}'_2, z)(H_2 - H'_2)_{vv} G(\mathcal{G}_2, z)_{vj}.$$

By multiplying both sides of (2.25) by $(|m_{sc}| q^{\mathrm{dist}_{\mathcal{G}_0}(i,j)})^{-1}$, we obtain

$$|G_{ij}(\mathcal{G}_2, z)| \left(|m_{sc}| q^{\mathrm{dist}_{\mathcal{G}_0}(i,j)} \right)^{-1}$$

$$\leqslant \Gamma' + \sum_{v \in \mathbb{G}_0} \Gamma' \Gamma |(H_2 - H'_2)_{vv}| |m_{sc}| q^{\mathrm{dist}_{\mathcal{G}_0}(i,v) + \mathrm{dist}_{\mathcal{G}_0}(v,j) - \mathrm{dist}_{\mathcal{G}_0}(i,j)}$$

$$\leqslant \Gamma' + \frac{1}{d-1} \Gamma' \Gamma \sum_{v \in \mathbb{G}_0} g(v) \leqslant 2^{\omega+1} + \frac{8\omega 2^{\omega+1}}{d-1} \Gamma \leqslant 2^{\omega+1} + \Gamma/2,$$

where the first inequality uses the triangle inequality $\mathrm{dist}_{\mathcal{G}_0}(i,v) + \mathrm{dist}_{\mathcal{G}_0}(v,j) - \mathrm{dist}_{\mathcal{G}_0}(i,j) \geqslant 0$ and $q \leqslant 1$, and the second and third inequalities follow from the assumptions $\sum g(v) \leqslant 8\omega$, $\sqrt{d-1} \geqslant 2^{\omega+2}$, and $\omega \geqslant 6$. By taking the maximum on the left-hand side of the above inequality and rearranging it, we get $\Gamma \leqslant 2^{\omega+2}$. $\qquad \square$

Proof of (2.14) *for* $g \not\equiv 0$. The extension to the case $g \not\equiv 0$ again follows by comparing to the case $g \equiv 0$. We define $\mathcal{H}_2$ and $\mathcal{H}'_2$ analogously to $\mathcal{G}'_2$ and $\mathcal{G}'_2$. Our goal now

is to bound

$$\Gamma := \max_{i,j \in \mathbb{H}_0} |G_{ij}(\mathcal{G}_2) - G_{ij}(\mathcal{H}_2)| \left(|m_{sc}|q^{\ell(i,j)+1}\right)^{-1}.$$

The resolvent identity (B.1) and (2.24) imply

(2.26) $$G_{ij}(\mathcal{G}'_2) - G_{ij}(\mathcal{G}_2) = \sum_{v \in \mathbb{G}_0} G_{iv}(\mathcal{G}'_2) \frac{m_{sc}g(v)}{d-1} G_{vj}(\mathcal{G}_2),$$

(2.27) $$G_{ij}(\mathcal{H}'_2) - G_{ij}(\mathcal{H}_2) = \sum_{v \in \mathbb{H}_0} G_{iv}(\mathcal{H}'_2) \frac{m_{sc}g(v)}{d-1} G_{vj}(\mathcal{H}_2).$$

For vertices $i, j \in \mathbb{H}_0$, set

$$\ell(i,j) := \max\{\ell : \text{all paths in } \mathcal{G}_0 \text{ from } i \text{ to } j \text{ of length} \leqslant \ell \text{ are contained in } \mathcal{H}_0\},$$

and given any $v \in \mathbb{G}_0$, we abbreviate $\ell = \ell(i,j)$, $\ell_1 = \text{dist}_{\mathcal{G}_0}(i,v)$, $\ell_2 = \text{dist}_{\mathcal{G}_0}(v,j)$. To bound Γ, we distinguish two cases:

(i) $\ell_1 + \ell_2 \geqslant \ell + 1$. Then already (2.12) implies

$$\left|G_{iv}(\mathcal{G}'_2) \frac{m_{sc}g(v)}{d-1} G_{vj}(\mathcal{G}_2)\right|, \left|G_{iv}(\mathcal{H}'_2) \frac{m_{sc}g(v)}{d-1} G_{vj}(\mathcal{H}_2)\right| \leqslant \frac{2^{2\omega+4}g(v)}{d-1} |m_{sc}|q^{\ell+1}.$$

(ii) $\ell_1 + \ell_2 \leqslant \ell$. Then by assumption we must have $v \in \mathbb{H}_0$, and $\ell(i,v) \geqslant \ell - \ell_2$ and $\ell(v,j) \geqslant \ell - \ell_1$. Therefore, using the case $g \equiv 0$ for $|G_{iv}(\mathcal{G}'_2) - G_{iv}(\mathcal{H}'_2)|$ and (2.12) for $|G_{vj}(\mathcal{G}_2)|$ and $|G_{iv}(\mathcal{H}'_2)|$,

$$\left|G_{iv}(\mathcal{G}'_2) \frac{m_{sc}g(v)}{d-1} G_{vj}(\mathcal{G}_2) - G_{iv}(\mathcal{H}'_2) \frac{m_{sc}g(v)}{d-1} G_{vj}(\mathcal{H}_2)\right|$$
$$\leqslant \frac{|m_{sc}|g(v)}{d-1} \left(|G_{iv}(\mathcal{G}'_2) - G_{iv}(\mathcal{H}'_2)||G_{vj}(\mathcal{G}_2)| + |G_{iv}(\mathcal{H}'_2)||G_{vj}(\mathcal{G}_2) - G_{vj}(\mathcal{H}_2)|\right)$$
$$\leqslant \frac{2^{\omega+2}(\Gamma + 2^{2\omega+2})g(v)}{d-1} |m_{sc}|q^{\ell+1}.$$

Taking the difference of (2.26) and (2.27), dividing both sides by $|m_{sc}|q^{\ell(i,j)+1}$, and then taking the maximum over $i, j \in \mathbb{H}_0$, this leads to

$$\Gamma \leqslant 2^{2\omega+2} + \frac{2^{\omega+2}(\Gamma + 2^{2\omega+2})}{d-1} \sum_{v \in \mathbb{G}_0} g(v).$$

Since by assumptions $\sum g(v) \leqslant 8\omega$, $\sqrt{d-1} \geqslant 2^{\omega+2}$ and $\omega \geqslant 6$, again rearranging the above expression, we get $\Gamma \leqslant 2^{2\omega+3}$. This finishes the proof. $\qquad \square$

3. Spectral Density and Eigenvectors

Recall that $\mathsf{G}_{N,d}$ denotes the set of simple d-regular graphs on the vertex set $[\![N]\!]$. Throughout the paper, we control error estimates in terms of (large powers of) the parameter

$$(3.1) \qquad\qquad q(z) := \frac{|m_{sc}(z)|}{\sqrt{d-1}} \leqslant \frac{1}{\sqrt{d-1}},$$

where $z \in \mathbb{C}_+$. We will often omit the parameter z from the notation if it is clear from the context.

The main result we will prove in this section is the following theorem, Theorem 3.1. It states that, in $\mathcal{D}$, the Green's function $G_{ij}(\mathcal{G})$ is well approximated by $P_{ij}(\mathcal{E}_{r_*}(i, j, \mathcal{G}))$, which is *random*, but only depends on the *local graph structure* of $\mathcal{G}$ near the vertices i and j. Together with this information on the local graph structure, the result of Theorem 3.1 implies the results stated in Section 1.1.

Theorem 3.1. *Fix* $\alpha > 4$, $\omega \geqslant 8$ *and* $\sqrt{d-1} \geqslant (\omega + 1)2^{2\omega+45}$, *and set* $\ell_* = \lfloor \alpha \log_{d-1} \log N \rfloor$ *and* $r_* = 2\ell_* + 1$. *Then, for* $\mathcal{G}$ *chosen uniformly from* $\mathsf{G}_{N,d}$, *the Green's function satisfies*

$$(3.2) \qquad\qquad |G_{ij}(\mathcal{G}, z) - P_{ij}(\mathcal{E}_{r_*}(i, j, \mathcal{G}), z)| \leqslant |m_{sc}(z)|q(z)^{r_*},$$

with probability $1 - \mathrm{o}(N^{-\omega + 8})$, *uniformly in* $i, j \in [\![N]\!]$, *and uniformly in* $z \in \mathcal{D}$, *where* $\mathcal{D}$ *is as in* (1.4). *Here we assume that* $N \geqslant N_0(\alpha, \omega, d)$ *is large enough and that* Nd *is even.*

We emphasize that, for fixed d, the right-hand side of (3.2) converges to 0, as $N \to \infty$, uniformly in the spectral domain $\mathcal{D}$. The constants in the statement of the theorem can be improved at the expense of a longer proof and a more complicated statement. We do not pursue this.

Remark 3.2. *The equation* (3.2) *implies that the* individual *entries of the Green's function do* not *concentrate. For example,*

$$G_{ii}(z) = P_{ii}(\mathcal{E}_{r_*}(i, i, \mathcal{G}), z) + \mathrm{O}(\log N)^{-\alpha}$$

and the first term on the right-hand side can be easily seen to depend strongly on the local graph structure. Its fluctuation is of order 1.

Proof of Theorem 1.1. Thanks local structure of a random regular graph, under the assumptions of the theorem, there are $\kappa > 0$ and $\delta > 0$ such that, with $R = \lfloor \kappa \log_{d-1} N \rfloor$, one can assume that the radius-R neighborhoods of all but N^δ many vertices of $\mathcal{G}$ coincide with those of the infinite d-regular tree, and that the R-neighborhoods of all other vertices have excess at most ω (see e.g. Proposition 2.1). Moreover, for the vertices i that have radius-R tree neighborhoods, we have (see e.g. Proposition 2.6)

$$(3.3) \qquad\qquad P_{ii}(\mathcal{E}_{r_*}(i, i, \mathcal{G})) = m_d.$$

The vertices whose R-neighbourhood has bounded excess still satisfy (see e.g. Proposition 2.7)

$$(3.4) \qquad\qquad |P_{ii}(\mathcal{E}_{r_*}(i, i, \mathcal{G}))| \leqslant 3|m_{sc}|/2 \leqslant 3/2.$$

Therefore (3.2) and (3.3) imply that $G_{ii}(z) = m_d(z) + O(|m_{sc}(z)|q(z)^{r_*})$ for all $z \in \mathcal{D}$ and at least $N - N^{-\delta}$ vertices $i \in [\![N]\!]$. For the remaining vertices, by (3.4), we still have $|G_{ii}(z)| \leqslant 2$. Thus

$$m(z) = \frac{1}{N} \sum_{i=1}^{N} G_{ii}(z) = m_d(z) + O(\log N)^{-\alpha},$$

as claimed. $\qquad\qquad\square$

3.1. **Proof outline and main ideas.** In this section, we give a high-level outline of the proof of Theorem 3.1, whose details occupy the remainder of the paper. The proof is based on the general principle that, for small distances, a random regular graph behaves almost deterministically, while on the other hand, for large distances, it behaves much like a random matrix.

3.1.1. *Parameters.* Throughout the paper, we fix constants $\alpha > 4$, $\omega \geqslant 8$, $0 < \delta < 1/\omega$, $0 < \kappa < \delta/(2\omega + 2)$, $\sqrt{d-1} \geqslant (\omega + 1)2^{2\omega+45}$, and set $\ell_* = \lceil \alpha \log_{d-1} \log N \rceil$ and $r_* = 2\ell_* + 1$. We also set $R = \lfloor \kappa \log_{d-1} N \rfloor$, and write $r = 2\ell + 1$, where ℓ is a parameter chosen such that

(3.5) $$\ell \in [\![\ell_*, 2\ell_*]\!].$$

We always assume that Nd is even and sufficiently large (depending on the previous parameters).

3.1.2. *Structure of the proof.* The proof consists of several sections, which we briefly describe in this section. Here, we also define several subsets of $\mathsf{G}_{N,d}$, namely the sets

$$\Omega^-(z, \ell) \subset \Omega(z, \ell) \subset \Omega_1^+(z, \ell) \subset \bar{\Omega} \subset \mathsf{G}_{N,d}, \quad \Omega_1'(z, \ell) \subset \bar{\Omega} \subset \mathsf{G}_{N,d}.$$

These sets depend on parameters $z \in \mathbb{C}_+$ and $\ell \in \mathbb{N}$ (and also on the previously fixed parameters).

Small distance structure; the set $\bar{\Omega}$. The small distance behavior is captured in terms of cycles in neighborhoods of radius R. For any graph, we define the *excess* to be the smallest number of edges that must be removed to yield a graph with no cycles (a forest). Then, with R, ω, δ as fixed above, we define the set $\bar{\Omega} \subset \mathsf{G}_{N,d}$ to consist of graphs such that

- the radius-R neighborhood of any vertex has excess at most ω;
- the number of vertices that have an R-neighborhood that contains a cycle is at most N^δ.

The set $\bar{\Omega}$ provides rough a priori stability at small distances. All regular graphs appearing throughout the paper will be members of $\bar{\Omega}$. It is well-known that $\mathbb{P}(\bar{\Omega}) \geqslant 1 - o(N^{-\omega+\delta})$; see Proposition 2.1.

Green's function approximation; the sets $\Omega(z, \ell)$ and $\Omega^-(z, \ell)$. For $z \in \mathbb{C}_+$, we define the set $\Omega(z, \ell) \subset \bar{\Omega}$ be the set of graphs $\mathcal{G}$ such that for any two vertices i, j in $[\![N]\!]$, it holds that

$$(3.6) \qquad |G_{ij}(z) - P_{ij}(\mathcal{E}_r(i, j, \mathcal{G}), z)| \leqslant |m_{sc}| q^r.$$

Our main goal is to prove that $\Omega(z, \ell)$ has high probability uniformly in the spectral domain $z \in \mathcal{D}$. That $\Omega(z, \ell)$ has high probability is not difficult to show if $|z|$ is large enough; see Section 3.2. To extend this estimate to smaller z, we define the set $\Omega^-(z, \ell) \subset \Omega(z, \ell)$ by the same conditions as $\Omega(z, \ell)$, except that the right-hand side in (3.6) is smaller by a factor $1/2$:

$$(3.7) \qquad |G_{ij}(z) - P_{ij}(\mathcal{E}_r(i, j, \mathcal{G}), z)| \leqslant \frac{1}{2}|m_{sc}| q^r.$$

Our main goal is to show that, for any $z \in \mathcal{D} \cap \Lambda_\ell$ (where the spectral domain is defined in (1.4) and Λ_ℓ is defined in (3.239)), if $\Omega(z, \ell)$ has high probability, then the event $\Omega(z, \ell) \setminus \Omega^-(z, \ell)$ has very small probability, so that $\Omega^-(z, \ell)$ still has high probability. Then, by the Lipschitz-continuity of the Green's function, it follows that

$\Omega^-(z, \ell) \subset \Omega(z', \ell)$ for small $|z - z'|$, and thus that $\Omega(z', \ell)$ also has high probability. This can then be repeated to show that $\Omega(z, \ell)$ holds for all $z \in \mathcal{D} \cap \Lambda_\ell$ with high probability. Since these sets Λ_ℓ all together cover $\mathcal{D}$, it follows that $\Omega(z, \ell_*)$ holds for all $z \in \mathcal{D}$ with high probability.

Local resampling. To show that $\Omega(z, \ell) \setminus \Omega^-(z, \ell)$ has small probability, we use the random matrix-like structure of random regular graphs at large distances. To this end, we fix a vertex, without loss of generality chosen to be 1, and abbreviate the ℓ-neighborhood of 1 (as a set of vertices in $[\![N]\!]$ and as a graph, respectively) by

$$(3.8) \qquad \mathbb{T} = \mathbb{B}_\ell(1, \mathcal{G}), \quad \mathcal{T} = \mathcal{B}_\ell(1, \mathcal{G}).$$

In Section 3.3, we resample the boundary of the neighborhood $\mathcal{T}$ by switching the boundary edges with uniformly chosen edges from the remainder of the graph. The switched graph is often denoted by $\tilde{\mathcal{G}}$. On the vertex set $\mathbb{T}$, it coincides with the unswitched graph $\mathcal{G}$, but the boundary of $\mathcal{T}$ in the switched graph $\tilde{\mathcal{G}}$ is now essentially random compared to the original graph $\mathcal{G}$.

Given $\mathcal{G}$, the switching is specified by the resampling data $\mathbf{S}$, which consists of μ independently chosen oriented edges from $\mathcal{G}^{(\mathbb{T})}$. The local resampling is implemented by switching a boundary edge of $\mathcal{T}$ with one of the independently chosen edges encoded by $\mathbf{S}$. In fact, in this operation, not all pairs of edges can be switched (are *switchable*) while keeping the graph simple. Therefore, given $\mathbf{S}$, we denote by $W_{\mathbf{S}} \subset [\![1, \mu]\!]$ the index set for switchable edges (see Section 3.3 for the definition), whose switching leaves the uniform measure on $\mathsf{G}_{N,d}$ invariant. For notational convenience, without loss of generality, we assume that $W_{\mathbf{S}} = \{1, 2, 3, \ldots, \nu\}$ where $\nu \leqslant \mu$ throughout the paper (except in the definition in Section 3.3).

Switching from $\mathcal{G}$ to $\tilde{\mathcal{G}}$. Throughout Sections 3.3.7–3.10, we condition on a graph $\mathcal{G}$ that satisfies certain estimates, and only use the randomness of the switching that specifies how to modify $\mathcal{G}$ to $\tilde{\mathcal{G}}$. By our choice of ℓ and using $\mathcal{T}$ has bounded excess

(which we can and do assume), the number of edges in the boundary of $\mathcal{T}$ is about $(\log N)^{O(1)}$. The randomness of these edges ultimately provides access to concentration estimates, which exhibit the random matrix-like structure of the random regular graph at large distances.

Note that, if we remove the vertex set $\mathbb{T}$ from $\mathcal{G}$, our switchings have a simpler effect than in $\mathcal{G}$: they only consist of removing the edges $\{b_i, c_i\}$ and adding instead $\{a_i, b_i\}$, for $i \in W_{\mathbf{S}}$. Therefore, instead of studying the change from $\mathcal{G}$ to $\tilde{\mathcal{G}}$ at once, it will be convenient to analyze the effect of the switching in several steps. For this, we define the following graphs (which need not be regular).

- $\mathcal{G}$ is the original unswitched graph;
- $\mathcal{G}^{(\mathbb{T})}$ is the unswitched graph with vertices $\mathbb{T}$ removed;
- $\hat{\mathcal{G}}^{(\mathbb{T})}$ is the intermediate graph obtained from $\mathcal{G}^{(\mathbb{T})}$ by removing the edges $\{b_i, c_i\}$ with $i \in W_{\mathbf{S}}$;
- $\tilde{\mathcal{G}}^{(\mathbb{T})}$ is the switched graph obtained from $\hat{\mathcal{G}}^{(\mathbb{T})}$ by adding the edges $\{a_i, b_i\}$ with $i \in W_{\mathbf{S}}$; and
- $\tilde{\mathcal{G}}$ is the switched graph $T_{\mathbf{S}}(\mathcal{G})$ (including vertices $\mathbb{T}$).

Following the conventions of Section 2.3, the deficit functions of these graphs are given by $d - \deg$, where deg the degree function of the graph considered, and we abbreviate their Green's functions by G, $G^{(\mathbb{T})}$, $\hat{G}^{(\mathbb{T})}$, $\tilde{G}^{(\mathbb{T})}$, and $\tilde{G}$ respectively.

Distance estimates. To use the local resampling, we require some estimates on the local distance structure of graphs and some a priori estimates on their Green's functions. These are collected in Sections 3.3.7–3.4. In fact, we use both the usual graph distance (of the unswitched and switched graphs) and a notion of "distance" that is defined in terms of the size of the Green's function of the graph from which the set $\mathbb{T}$ is removed (again for the unswitched and switched graph).

The need for the Green's function distance arises as follows. While estimates that involve sums over the diagonal of the Green's function can be controlled quite well

using only the graph distance, estimates of sums of off-diagonal terms are more deli-
cate because the number terms is *squared* compared to the diagonal terms. By direct
combinatorial arguments, it would be difficult to control large distances sufficiently
precisely. However, to understand spectral properties, it is the size of the Green's
function rather than distances themselves that is relevant; and while the size of the
Green's function between two vertices is directly related to the distance between them
if there are only few cycles, on a global scale (where many cycles could be present)
cancellations can make the Green's function much smaller. These cancellations are
captured in terms of a Ward identity, which states that the Green's function of any
symmetric matrix obeys (see also Appendix B)

$$(3.9) \qquad \frac{1}{N} \sum_{i=1}^{N} |G_{ij}(z)|^2 = \frac{\operatorname{Im} G_{ii}(z)}{\operatorname{Im} z}.$$

Removing the neighborhood $\mathbb{T}$ and stability under resampling; the sets $\Omega_1^+(z, \ell)$. Our
goal is to show that estimates on the Green's function of $\mathcal{G}$ improve near the vertex
1 under the above mentioned local resampling. For this, we work with the Green's
function of the graph $\mathcal{G}^{(\mathbb{T})}$ obtained from $\mathcal{G}$ by removing the vertex set $\mathbb{T}$ (on which
the graph does not change under switching).

As a preliminary step to showing that the estimates for the Green's function im-
prove, we show that they are stable under the operation of removing $\mathbb{T}$ and resam-
pling, i.e., roughly that the estimates analogous to those assumed continue to hold.
More precisely, in Section 3.5, we show that if $\mathcal{G} \in \Omega(z, \ell)$, then the (non-regular)
graph $\mathcal{G}^{(\mathbb{T})}$ obeys the analogous estimate

$$(3.10) \qquad \left| G_{ij}(\mathcal{G}^{(\mathbb{T})}, z) - P_{ij}(\mathcal{E}_r(i, j, \mathcal{G}^{(\mathbb{T})}), z) \right| \leqslant 2|m_{sc}|q^r.$$

We define the set $\Omega_1^+(z, \ell) \subset \bar{\Omega}$ similarly as the set $\Omega(z, \ell)$, except that $\mathcal{G}$ is replaced
by the graph $\mathcal{G}^{(\mathbb{T})}$ (and with different constant), i.e., $\Omega_1^+(z, \ell)$ is the set of $\mathcal{G} \in \bar{\Omega}$ such

that

(3.11) $$\left| G_{ij}(\mathcal{G}^{(\mathbb{T})}, z) - P_{ij}(\mathcal{E}_r(i, j, \mathcal{G}^{(\mathbb{T})}), z) \right| \leqslant 2^{10} |m_{sc}| q^r.$$

Clearly, by (3.10), we have $\Omega(z, \ell) \subset \Omega_1^+(z, \ell)$. In Section 3.6, we show that if $\mathcal{G}^{(\mathbb{T})}$ obeys the (stronger) estimate (3.10), then with high probability the resampled graph obeys $\mathcal{G}^{(\mathbb{T})} \in \Omega^+(z, \ell)$.

Locally improved Green's function approximation; the sets $\Omega_1'(z, \ell)$. The set $\Omega_1'(z, \ell) \subset \bar{\Omega}$ is defined by the improved estimates (3.212)–(3.215) near the vertex 1, with constant $K = 2^{10}$. In Sections 3.7–3.10, it is proved that if we start with a graph $\mathcal{G} \in \Omega_1^+(z, \ell)$, with high probability with respect to the local resampling around vertex 1, the switched graph $\tilde{\mathcal{G}}$ belongs to $\Omega_1'(z, \ell)$.

Involution. To sum up, the argument outlined above shows that, for any graph $\mathcal{G}$ in $\Omega(z, \ell)$, with high probability with respect to the randomness of the local resampling, the switched graph $\tilde{\mathcal{G}}$ is in the set $\Omega_1'(z, \ell)$. However, our goal was to show that a uniform d-regular graph $\mathcal{G}$ is in $\Omega_1'(z, \ell)$, except for an event of small probability. This follows from the statement we proved for $\tilde{\mathcal{G}}$ using that our switching acts as an involution on the larger product probability space (see Proposition 3.9).

Self-consistent equation. The sets $\Omega_1^+(z, \ell)$ and $\Omega_1'(z, \ell)$ depend on the choice of vertex 1. However, for any $i \in [\![N]\!]$, we can define $\Omega_i'(z, \ell)$ in the same way, by replacing the vertex 1 in the above definitions by vertex i (or using symmetry). By a union bound, then also the union of the events $\Omega_i'(z, \ell)$ over $i \in [\![N]\!]$ holds with high probability. On the latter event, we derive (in Section 3.11) a *self-consistent equation* for the quantity

$$Q(\mathcal{G}) = \frac{1}{Nd} \sum_{(i,j) \in \vec{E}} G_{jj}^{(i)}(\mathcal{G}),$$

where the sum ranges over the set of oriented edges in $\mathcal{G}$, and $G^{(i)}(\mathcal{G})$ is the Green's function of the graph $\mathcal{G}$ with vertex i removed. On the infinite d-regular tree, it is

straightforward computation to show that $G_{jj}^{(i)}(z) = m_{sc}(z)$ holds for any directed edge (i, j) (see Proposition 2.6). For the random regular graph, we will show that $Q(\mathcal{G})$ obeys (see (3.230))

$$Q(\mathcal{G}) - m_{sc} = \frac{d-2}{d-1} m_d m_{sc}^{2\ell+1}(Q(\mathcal{G}) - m_{sc}) + \text{error}.$$

The main result of Section 3.11, proved using this self-consistent equation, is that, for any $z \in \mathcal{D} \cap \Lambda_\ell$,

$$\bigcap_{1 \leqslant i \leqslant N} \Omega_i'(z, \ell) \subset \Omega^-(z, \ell),$$

where $\Lambda_\ell \subset \mathbb{C}_+$ is a domain on which the self-consistent equation is not singular (see Section 3.11 for details). In the final step, we will use different choices of ℓ to cover the entire spectral domain $\mathcal{D}$.

Conclusion. In summary, in Sections 3.5–3.11, we show that the probability of $\Omega(z, \ell) \backslash \Omega^-(z, \ell)$ is negligible. By the Lipschitz property of the Green's function, $\Omega^-(z, \ell) \subset \Omega(z', \ell)$ given that $|z - z'|$ is small enough. It follows that if $\Omega^-(z, \ell)$ holds with high probability, then $\Omega^-(z, \ell) \cap \Omega^-(z', \ell)$ holds with high probability. This can then be repeated to show that $\Omega(z, \ell)$ holds for all $z \in \mathcal{D} \cap \Lambda_\ell$ with high probability. The proof of Theorem 3.1 is then completed by showing that $\mathcal{D} \subset \cup_{\ell \in [\![\ell_*, 2\ell_*]\!]} \Lambda_\ell$ and thus $\Omega(z, \ell_*)$ holds for all $z \in \mathcal{D}$ with high probability.

3.2. **Initial estimates.** As the first step of the proof of Theorem 3.1, we now show that (3.2) holds whenever $|z| \geqslant 2d - 1$. Indeed, the following proposition states that (3.2) holds deterministically for $|z| \geqslant 2d - 1$ under the assumption that the graph has locally bounded excess, which is guaranteed to hold with high probability by (2.6). (Related results appear in [33].)

Proposition 3.3. *Let* $\omega \geqslant 6$, $\sqrt{d-1} \geqslant 2^{\omega+2}$ *and* $N \geqslant N_0(\omega, d)$ *large enough. Let* $\mathcal{G}$ *be a d-regular graph on* N *vertices, with excess at most* ω *in any radius-R neighborhood. Then for any* $z \in \mathbb{C}_+$ *with* $|z| \geqslant 2d - 1$, *and any* $i, j \in \mathbb{G}$, *the Green's*

function of $\mathcal{G}$ *satisfies*

$$(3.12) \qquad |G_{ij}(z) - P_{ij}(\mathcal{E}_r(i,j,\mathcal{G}),z)| \leqslant \frac{1}{2}|m_{sc}|q^r.$$

3.2.1. *Proof of Proposition 3.3.* To prove Proposition 3.3, we need an upper bound on the entries of the Green's function. It can be obtained, for example, by the Combes–Thomas method [27].

Lemma 3.4. *For any finite simple graph* $\mathcal{G}$ *with degree bounded by* d*, and any* z *with* $|z| \geqslant 2d - 1$*,*

$$(3.13) \qquad |G_{ij}(z)| \leqslant \frac{1}{d}, \qquad |G_{ij}(z)| \leqslant \frac{1}{(d-1)^{\mathrm{dist}_\mathcal{G}(i,j)/2}}.$$

Proof. We denote the normalized adjacency matrix of $\mathcal{G}$ by H (where we recall that the normalization of the entries is always by $1/\sqrt{d-1}$). The first bound in (3.13) is immediate since the spectrum of H is contained in $[-d/\sqrt{d-1}, d/\sqrt{d-1}]$, which implies that

$$(3.14) \qquad |G_{ij}(z)| \leqslant \frac{1}{|z| - d/\sqrt{d-1}} \leqslant \frac{1}{d}.$$

To show the second bound, set $\tau = \frac{1}{2}\log(d-1)$. Fix a vertex i, and define the diagonal matrix M by

$$M_{jj} = \exp\{\tau \mathrm{dist}_\mathcal{G}(i,j)\}.$$

Then we have

$$G_{ij}e^{\tau \mathrm{dist}_\mathcal{G}(i,j)} = \langle \delta_j, MGM^{-1}\delta_i \rangle = \langle \delta_j, (MHM^{-1} - z)^{-1}\delta_i \rangle.$$

The entries of the matrix MHM^{-1} are given by

$$(MHM^{-1})_{xy} = e^{\tau(\mathrm{dist}_\mathcal{G}(i,x) - \mathrm{dist}_\mathcal{G}(i,y))}H_{xy}.$$

If $H_{xy} \neq 0$, then $|\text{dist}_{\mathcal{G}}(i,x) - \text{dist}_{\mathcal{G}}(i,y)| \leqslant 1$, and

$$\max_x \sum_y |(MHM^{-1})_{xy}| \leqslant de^\tau/\sqrt{d-1} \leqslant d,$$

$$\max_y \sum_x |(MHM^{-1})_{xy}| \leqslant de^\tau/\sqrt{d-1} \leqslant d.$$

Therefore $\|MHM^{-1}\|_{\infty\to\infty}$ and $\|MHM^{-1}\|_{1\to1}$ are bounded by d, and by interpolation

$$\|MHM^{-1}\|_{2\to2} \leqslant \sqrt{\|MHM^{-1}\|_{1\to1}\|MHM^{-1}\|_{\infty\to\infty}} \leqslant d.$$

Therefore, the spectrum of MHM^{-1} is contained in the set $\{z \in \mathbb{C} : |z| \leqslant d\}$. In particular, for z such that $|z| \geqslant 2d - 1$, its distance to the spectrum of MHM^{-1} is at least 1, and thus

$$|G_{ij}e^{\tau\text{dist}_{\mathcal{G}}(i,j)}| = |\langle \delta_j, (MHM^{-1} - z)^{-1}\delta_i\rangle| \leqslant 1,$$

which implies (3.13). This completes the proof. $\qquad\square$

Proof of Proposition 3.3. Let $r_0 := \lceil r + 1 - 2(r+2)\log_{d-1}|m_{sc}| \rceil = \mathrm{O}(r)$. Then, for vertices i, j such that $\text{dist}_{\mathcal{G}}(i,j) \geqslant r_0$, Lemma 3.4 implies

$$|G_{ij}(z)| \leqslant \frac{1}{(d-1)^{r_0/2}} \leqslant |m_{sc}|q^{r+1},$$

and in particular (3.12) follows since $q \leqslant 1/\sqrt{d-1} \leqslant 1/2$ and $P_{ij}(\mathcal{E}_r(i,j,\mathcal{G})) = 0$.

Thus we can assume $\text{dist}_{\mathcal{G}}(i,j) < r_0$. Let $\mathcal{G}_0 := \mathcal{B}_{r_0+r}(i,\mathcal{G})$, let $\mathcal{G}_1 = \text{TE}(\mathcal{G}_0)$ be the tree extension of $\mathcal{G}_0$, and let P be the Green's function of $\mathcal{G}_1$. Then, by (2.14), we have

$$(3.15) \qquad |P_{ij} - P_{ij}(\mathcal{E}_r(i,j,\mathcal{G}))| \leqslant 2^{2\omega+3}|m_{sc}|q^{r+1}.$$

Therefore it suffices to prove the claim with $P_{ij}(\mathcal{E}_r(i,j,\mathcal{G}))$ replaced by P_{ij}, and an additional factor $1/2$ on the right-hand side. Let $\mathbb{T}_0 := \mathbb{B}_{r_0}(i,\mathcal{G})$ and $\partial\mathbb{T}_0 = \{v \in \mathcal{G} : \text{dist}_\mathcal{G}(v,\mathbb{T}_0) = 1\}$. By the Schur complement formula (B.4),

$$G|_{\mathbb{T}_0} = (H - z - B'G^{(\mathbb{T}_0)}B)^{-1},$$

$$P|_{\mathbb{T}_0} = (H - z - B'P^{(\mathbb{T}_0)}B)^{-1},$$

where H is the normalized adjacency matrix on $\mathbb{T}_0$ induced by $\mathcal{G}$ and B is the part of the adjacency matrix of the edges from $\partial\mathbb{T}_0$ to $\mathbb{T}_0$. Taking the difference of the last two equations, for any $i,j \in \mathbb{T}_0$,

$$|(G-P)_{ij}| \leqslant \sum_{x,y \in \partial\mathbb{T}_0} |(PB')_{ix}| \left(|G_{xy}^{(\mathbb{T}_0)}| + |P_{xy}^{(\mathbb{T}_0)}|\right) |(BG)_{yj}|.$$

Since the radius-R neighborhood of i has excess at most ω, each row of B contains at most $\omega + 1$ nonzero entries. Therefore, by (2.12), Lemma 3.4, and noticing that $\text{dist}_\mathcal{G}(i,x) \geqslant r_0 + 1$ and $\text{dist}_\mathcal{G}(y,j) \geqslant r_0 + 1 - \text{dist}_\mathcal{G}(i,j)$, we have

$$|(PB')_{ix}| \leqslant 2^{\omega+2}(\omega+1)q^{r_0+1}, \quad |(BG)_{yj}| \leqslant \frac{\omega+1}{(d-1)^{(r_0+1-\text{dist}_\mathcal{G}(i,j))/2}},$$

where we recall the definition $q = |m_{sc}|/\sqrt{d-1}$. Moreover, it follows from (3.29) that

$$|\{x \in \partial\mathbb{T}_0 : \text{dist}_{\mathcal{G}^{(\mathbb{T}_0)}}(x, \partial\mathbb{T}_0 \setminus \{x\}) \leqslant R/2\}| \leqslant 2\omega,$$

using that $R > 2r_0$. Therefore, by the second bound of (3.13), $|G_{xy}^{(\mathbb{T}_0)}| \leqslant (d-1)^{-R/4}$ for all $x,y \in \partial\mathbb{T}_0$ except for the diagonal entries and at most $4\omega^2$ off-diagonal entries. By the first bound of (3.13), for these remaining entries we have $|G_{xy}^{(\mathbb{T}_0)}| \leqslant 1/d$. The

same bounds hold for $P^{(\mathbb{T}_0)}$ instead of $G^{(\mathbb{T}_0)}$. As a result, we obtain

$$|(G - P)_{ij}| \leqslant \frac{2^{\omega+2}(\omega+1)^2 q^{r_0+1}}{(d-1)^{(r_0-\text{dist}_\mathcal{G}(i,j)+1)/2}} \sum_{x,y \in \partial\mathbb{T}_0} \left(|(G_{xy}^{(\mathbb{T}_0)}| + |P^{(\mathbb{T}_0)})_{xy}|\right)$$

$$\leqslant \frac{2^{\omega+3}(\omega+1)^2|m_{sc}|^{r_0+1}}{(d-1)^{r_0+1-\text{dist}_\mathcal{G}(i,j)/2}} \left(\frac{|\partial\mathbb{T}_0| + 4\omega^2}{d} + \frac{|\partial\mathbb{T}_0|^2}{(d-1)^{R/4}}\right).$$

Using that $|\partial\mathbb{T}_0| \leqslant d(d-1)^{r_0}$, that $|m_{sc}| \leqslant 1/d$ for $|z| \geqslant 2d-1$, that $d-1 \geqslant 2(\omega+1)$, as well as that $R > 4r_0$, the right-hand side is bounded by

$$|(G - P)_{ij}| \leqslant \frac{2^{\omega+3}(\omega+1)^2|m_{sc}|^{r_0+1}}{(d-1)^{r_0+1-\text{dist}_\mathcal{G}(i,j)/2}} \left(\frac{d(d-1)^{r_0} + 4\omega^2}{d} + \frac{d^2(d-1)^{2r_0}}{(d-1)^{R/4}}\right)$$

$$\leqslant \frac{2^{\omega+4}(\omega+1)^2}{(d-1)^{r_0+2-\text{dist}_\mathcal{G}(i,j)/2}} \leqslant \frac{2^{\omega+2}}{(d-1)^{r_0/2}} \leqslant 2^{\omega+2}|m_{sc}|q^{r+1},$$

where we used that $\text{dist}_\mathcal{G}(i,j) < r_0$. Together with (3.15), we conclude that

$$(3.16) \qquad |G_{ij}(z) - P_{ij}(\mathcal{E}_r(i,j,\mathcal{G}),z)| \leqslant (2^{2\omega+3} + 2^{\omega+2})|m_{sc}|q^{r+1} \leqslant \frac{1}{2}|m_{sc}|q^r,$$

where the last inequality follows from $q = |m_{sc}|/\sqrt{d-1} \leqslant 2^{-3(\omega+2)}$, using that $\sqrt{d-1} \geqslant 2^{\omega+2}$. $\qquad\square$

3.3. **Local resampling by switching.** In this section, we define a *local resampling* of a random regular graph by using switchings. We effectively resample the edges on the boundary of balls of radius ℓ, by switching them with random edges from the remainder of the graph. This resampling generalizes the local resampling introduced in [19], where switchings were used to resample the neighbors of a vertex (corresponding to $\ell = 0$). The local resampling provides an effective access to the randomness of the random regular graph, which is fundamental for the remainder of the paper.

3.3.1. *Definitions.* To introduce the local resampling, we require some definitions. Graphs and edges. We consider simple d-regular graphs on vertex set $[\![N]\!]$ and identify such graphs with their sets of edges throughout this section. (Deficit functions do not play a role in this section.) For any graph $\mathcal{G}$, we denote the set of unoriented edges

by E, and the set of oriented edges by $\vec{E} := \{(u,v), (v,u) : \{u,v\} \in E\}$. For a subset $\vec{S} \subset \vec{E}$, we denote by S the set of corresponding non-oriented edges. For a subset $S \subset E$ of edges we denote by $[S] \subset [\![N]\!]$ the set of vertices incident to any edge in S. Moreover, for a subset $\mathbb{V} \subset [\![N]\!]$ of vertices, we define $E|_{\mathbb{V}}$ to be the subgraph of $\mathcal{G}$ induced on $\mathbb{V}$.
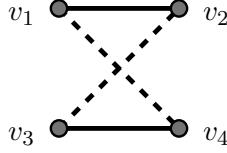

FIGURE 4. The switching encoded by the two directed edges $\vec{S} = \{(v_1, v_2), (v_3, v_4)\}$ replaces the unoriented edges $\{v_1, v_2\}, \{v_3, v_4\}$ by $\{v_1, v_4\}, \{v_2, v_3\}$.

Switchings. A (simple) switching is encoded by a pair of oriented edges $\vec{S} = \{(v_1, v_2), (v_3, v_4)\} \subset \vec{E}$. We assume that the two edges are disjoint, i.e. that $|\{v_1, v_2, v_3, v_4\}| = 4$. Then the switching consists of replacing the edges $\{v_1, v_2\}, \{v_3, v_4\}$ by the edges $\{v_1, v_4\}, \{v_2, v_3\}$, as illustrated in Figure 4. We denote the graph after the switching $\vec{S}$ by $T_{\vec{S}}(\mathcal{G})$, and the new edges $\vec{S}' = \{(v_1, v_4), (v_2, v_3)\}$ by

$$(3.17) \qquad\qquad\qquad T(\vec{S}) = \vec{S}'.$$

(Double switchings, which we used in [19], are not needed in this paper; henceforth we will therefore refer to simple switchings as switchings.)

Resampling data. Our local resampling involves a center vertex, which by symmetry we now assume to be 1, and a radius ℓ. Given a d-regular graph $\mathcal{G}$, we abbreviate $\mathbb{T} = \mathbb{B}_\ell(1, \mathcal{G})$ and $\mathcal{T} = \mathcal{B}_\ell(1, \mathcal{G})$. The edge boundary $\partial_E \mathcal{T}$ of $\mathcal{T}$ consists of the edges in $\mathcal{G}$ with one vertex in $\mathbb{T}$ and the other vertex in $[\![N]\!] \setminus \mathbb{T}$, as illustrated in Figure 5. Our local resampling switches the edge boundary of $\mathcal{T}$ with randomly chosen edges in $\mathcal{G}^{(\mathbb{T})}$ if the switching is admissible (see below), and leaves them in place otherwise.
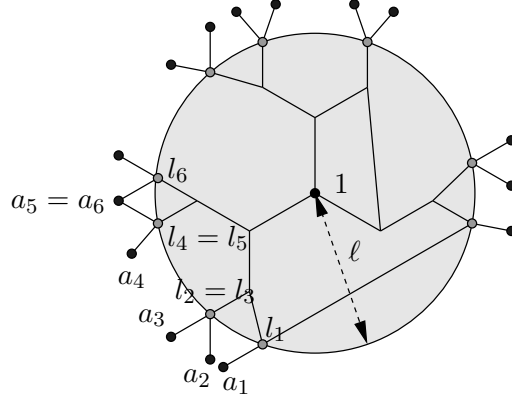
FIGURE 5. The figure illustrates the neighborhood $\mathcal{T} = \mathcal{B}_\ell(1, \mathcal{G})$ (within the shaded area) and its edge boundary $\partial_E\mathcal{T}$, consisting of the edges $e_i = \{l_i, a_i\}$, $1 \leqslant i \leqslant \mu$. Our local resampling switches the switchable edges e_i (corresponding to $i \in W_\mathbf{S}$) with randomly chosen edges from the remainder of the graph (not shown). Several exceptional cases can occur. In particular, the vertices a_i are not necessarily distinct (e.g., $a_5 = a_6$ in the figure), and the boundary vertices l_i may have different degrees in the graph obtained by removing the set $\mathcal{T}$ (e.g., l_1 has only one outgoing edge in the figure, while most of the other l_i has two outgoing edges).

To be precise, given a graph $\mathcal{G}$, we enumerate $\partial_E\mathcal{T}$ as $\partial_E\mathcal{T} = \{e_1, e_2, \ldots, e_\mu\}$, and orient the edges e_i by defining $\vec{e}_i$ to have the same vertices as e_i and to be directed from a vertex $l_i \in \mathbb{T}$ to a vertex $a_i \in [\![N]\!] \setminus \mathbb{T}$. The directed edges $\vec{e}_i = (l_i, a_i)$ are illustrated in Figure 5. Note that μ and the edges $e_1, \ldots, e_\mu$ depend on $\mathcal{G}$.

Then we choose $(b_1, c_1), \ldots, (b_\mu, c_\mu)$ to be independent, uniformly chosen oriented edges from the graph $\mathcal{G}^{(\mathbb{T})}$, i.e., the edges of $\mathcal{G}$ that are not incident to $\mathbb{T}$, and define

$$(3.18) \qquad \vec{S}_i = \{\vec{e}_i, (b_i, c_i)\}, \qquad \mathbf{S} = (\vec{S}_1, \vec{S}_2, \ldots, \vec{S}_\mu).$$

The sets $\mathbf{S}$ will be called the *resampling data* for $\mathcal{G}$. By definition, the edges e_i are distinct, but the vertices a_i are not necessarily distinct and neither are the vertices l_i.

Admissible switchings. For $i \in [\![1, \mu]\!]$, we define the indicator functions

$$I_i \equiv I_i(\mathcal{G}, \mathbf{S}) := \mathbf{1}(|[S_i]| = 4, E|_{[S_i]} = S_i),$$

$$J_i \equiv J_i(\mathcal{G}, \mathbf{S}) := \mathbf{1}(|[S_i] \cap [S_j]| \leqslant 1 \text{ for all } j \neq i),$$

and the set of *admissible switchings*

(3.19) $$W_{\mathbf{S}} \equiv W(\mathcal{G}, \mathbf{S}) := \{i \in [\![1, \mu]\!] : I_i(\mathcal{G}, \mathbf{S}) J_i(\mathcal{G}, \mathbf{S}) = 1\}.$$

The interpretation of $I_i = 1$ is that the graph $E|_{[S_i]}$ is 1-regular. The interpretation of $J_i = 1$ is that the edges of S_i do not interfere with the edges of any other S_j. Indeed, the condition $|[S_i] \cap [S_j]| \leqslant 1$ guarantees that the switchings encoded by $\vec{S}_i$ and $\vec{S}_j$ do not influence each other, meaning that $T_{\vec{S}_i}$ and $T_{\vec{S}_j}$ commute. We say that the index $i \in [\![1, \mu]\!]$ is *admissible* or *switchable* if $i \in W_{\mathbf{S}}$.

Let $\nu := |W_{\mathbf{S}}|$ be the number of admissible switchings and $i_1, i_2, \ldots, i_\nu$ be an arbitrary enumeration of $W_{\mathbf{S}}$. Then we define the switched graph by

(3.20) $$T_{\mathbf{S}}(\mathcal{G}) := \left(T_{\vec{S}_{i_1}} \circ \cdots \circ T_{\vec{S}_{i_\nu}} \right)(\mathcal{G})$$

and the switching data by

(3.21) $$T(\mathbf{S}) := (T_i(\vec{S}_i), \ldots, T_\mu(\vec{S}_\mu)), \quad T_i(\vec{S}_i) = \begin{cases} T(\vec{S}_i) & (i \in W_{\mathbf{S}}) \\ \vec{S}_i & (i \notin W_{\mathbf{S}}). \end{cases}$$

3.3.2. *Reversibility.* To make the structure more clear, we introduce an enlarged probability space. Equivalently to the definition above, the sets $\vec{S}_i$ are uniformly distributed over

$$\mathsf{S}_i(\mathcal{G}) = \{\vec{S} \subset \vec{E} : \vec{S} = \{\vec{e}_i, \vec{e}\}, \vec{e} \text{ is not incident to } \mathbb{T}\},$$

i.e., the set of pairs of oriented edges in $\vec{E}$ containing $\vec{e}_i$ and another oriented edge in $\mathcal{G}^{(\mathbb{T})}$. Therefore $\mathbf{S} = (\vec{S}_1, \vec{S}_2, \ldots, \vec{S}_\mu)$ is uniformly distributed over the set $\mathsf{S}(\mathcal{G}) = \mathsf{S}_1(\mathcal{G}) \times \cdots \times \mathsf{S}_\mu(\mathcal{G})$.

Definition 3.5. *For any graph $\mathcal{G} \in \mathsf{G}_{N,d}$, denote by $\iota(\mathcal{G}) = \{\mathcal{G}\} \times \mathsf{S}(\mathcal{G})$ the* fibre *of local resamplings of $\mathcal{G}$ (with respect to vertex 1), and define the enlarged probability space*

$$\tilde{\mathsf{G}}_{N,d} = \iota(\mathsf{G}_{N,d}) = \bigsqcup_{\mathcal{G} \in \mathsf{G}_{N,d}} \iota(\mathcal{G})$$

with the probability measure $\tilde{\mathbb{P}}(\mathcal{G}, \mathbf{S}) := \mathbb{P}(\mathcal{G})\mathbb{P}_\mathcal{G}(\mathbf{S}) = (1/|\mathsf{G}_{N,d}|)(1/|\mathsf{S}(\mathcal{G})|)$ for any $(\mathcal{G}, \mathbf{S}) \in \tilde{\mathsf{G}}_{N,d}$. Here $\mathbb{P}(\mathcal{G}) = 1/|\mathsf{G}_{N,d}|$ is the uniform probability measure on $\mathsf{G}_{N,d}$, and for $\mathcal{G} \in \mathsf{G}_{N,d}$, we denote the uniform probability measure on $\mathsf{S}(\mathcal{G})$ by $\mathbb{P}_\mathcal{G}$.

Let $\pi : \tilde{\mathsf{G}}_{N,d} \to \mathsf{G}_{N,d}$, $(\mathcal{G}, \mathbf{S}) \mapsto \mathcal{G}$ be the canonical projection onto the first component.

Proposition 3.6. *π is measure preserving: $\mathbb{P} = \tilde{\mathbb{P}} \circ \pi^{-1}$.*

Proof. Note that $\pi^{-1}(\mathcal{G}) = \iota(\mathcal{G})$. Therefore

$$(3.22) \qquad \tilde{\mathbb{P}}(\pi^{-1}(\mathcal{G})) = \tilde{\mathbb{P}}(\iota(\mathcal{G})) = \sum_{\mathbf{S} \in \mathsf{S}(\mathcal{G})} \tilde{\mathbb{P}}(\mathcal{G}, \mathbf{S}) = \mathbb{P}(\mathcal{G}) \sum_{\mathbf{S} \in \mathsf{S}(\mathcal{G})} \frac{1}{|\mathsf{S}(\mathcal{G})|} = \mathbb{P}(\mathcal{G}),$$

as claimed. $\square$

On the enlarged probability space, we define the maps

$$(3.23) \qquad\qquad \tilde{T} : \tilde{\mathsf{G}}_{N,d} \to \tilde{\mathsf{G}}_{N,d}, \quad \tilde{T}(\mathcal{G}, \mathbf{S}) := (T_\mathbf{S}(\mathcal{G}), T(\mathbf{S})),$$

$$(3.24) \qquad\qquad T : \tilde{\mathsf{G}}_{N,d} \to \mathsf{G}_{N,d}, \quad T(\mathcal{G}, \mathbf{S}) := \pi(\tilde{T}(\mathcal{G}, \mathbf{S})) = T_\mathbf{S}(\mathcal{G}).$$

For the statement of the next proposition, recall that $\mathsf{G}_{N,d}$ denotes the set of simple d-regular graph on $[\![N]\!]$. For any finite graph $\mathcal{T}$ on a subset of $[\![N]\!]$, we define

$\mathsf{G}_{N,d}(\mathcal{T}) := \{\mathcal{G} \in \mathsf{G}_{N,d} : \mathcal{B}_\ell(1, \mathcal{G}) = \mathcal{T}\}$ to be the set of d-regular graphs whose radius-ℓ neighborhood of the vertex i in $\mathcal{G}$ is $\mathcal{T}$.

Proposition 3.7. *For any graph $\mathcal{T}$, we have*

$$(3.25) \qquad\qquad \tilde{T}(\iota(\mathsf{G}_{N,d}(\mathcal{T}))) \subset \iota(\mathsf{G}_{N,d}(\mathcal{T})),$$

and $\tilde{T}$ is an involution: $\tilde{T} \circ \tilde{T} = \mathrm{id}$.

Proof. The first claim is obvious by construction. To verify that $\tilde{T}$ is an involution, let $(\mathcal{G}, \mathbf{S}) \in \tilde{\mathsf{G}}_{N,d}$ and abbreviate $(\tilde{\mathcal{G}}, \tilde{\mathbf{S}}) = \tilde{T}(\mathcal{G}, \mathbf{S})$. Then, due to (3.25), the edge boundaries of the ℓ-neighborhoods of 1 have the same number of edges μ in $\tilde{\mathcal{G}}$ and $\mathcal{G}$. Moreover, we can choose the (arbitrary) enumeration of the boundary of the ℓ-ball in $\tilde{\mathcal{G}}$ such that, for any $i \in [\![1, \mu]\!]$, we have $T_i(\vec{S}_i) \in \mathsf{S}_i(\tilde{\mathcal{G}})$. Define

$$\tilde{W}_{\tilde{\mathbf{S}}} \equiv W(\tilde{\mathcal{G}}, \tilde{\mathbf{S}}) := \{i \in [\![1, \mu]\!] : I_i(\tilde{\mathcal{G}}, \tilde{\mathbf{S}}) J_i(\tilde{\mathcal{G}}, \tilde{\mathbf{S}}) = 1\}.$$

We claim that $\tilde{W}_{\tilde{\mathbf{S}}} = W_{\mathbf{S}}$. First, by definition of switchings, we have $[T_i(S_i)] = [S_i]$ for any $i \in [\![1, \mu]\!]$. Thus $J_i(\tilde{\mathcal{G}}, \tilde{\mathbf{S}}) = J_i(\mathcal{G}, \mathbf{S})$, and it suffices to verify that $I_i(\tilde{\mathcal{G}}, \tilde{\mathbf{S}}) = I_i(\mathcal{G}, \mathbf{S})$ also holds for all $i \in [\![1, \mu]\!]$. For $i \notin W_{\mathbf{S}}$, the switching of S_i does not take place, i.e., $\tilde{\mathcal{G}}|_{[S_i]} = \mathcal{G}|_{[S_i]}$ and therefore $I_i(\tilde{\mathcal{G}}, \tilde{\mathbf{S}}) = I_i(\mathcal{G}, \mathbf{S})$. On the other hand, for $i \in W_{\mathbf{S}}$, the subgraph $\mathcal{G}|_{[S_i]}$ is 1-regular, i.e., $I_i(\mathcal{G}, \mathbf{S}) = 1$, and the other S_j with $j \in W_{\mathbf{S}}$ intersect S_i at most at one vertex. Therefore, $\tilde{\mathcal{G}}|_{[S_i]} = T_{\vec{S}_i} \mathcal{G}|_{[S_i]}$ and the graph $\tilde{\mathcal{G}}|_{[S_i]}$ is again 1-regular, i.e., $I_i(\tilde{\mathcal{G}}, \tilde{\mathbf{S}}) = 1$ as needed.

In summary, we have verified the claim $\tilde{W}_{\tilde{\mathbf{S}}} = W_{\mathbf{S}}$. By definition of our switchings, it follows that $T(\tilde{\mathbf{S}}) = \mathbf{S}$ and $T_{\tilde{\mathbf{S}}}(\tilde{\mathcal{G}}) = \mathcal{G}$. Therefore $\tilde{T}$ is an involution. $\qquad\square$

Proposition 3.8. *$\tilde{T}$ and T are measure preserving: $\tilde{\mathbb{P}} \circ \tilde{T}^{-1} = \tilde{\mathbb{P}}$ and $\tilde{\mathbb{P}} \circ T^{-1} = \mathbb{P}$.*

In other words, that T is measure preserving means that if $\mathcal{G}$ is uniform over $\mathsf{G}_{N,d}$, and given $\mathcal{G}$, we choose $\mathbf{S}$ uniform over $\mathsf{S}(\mathcal{G})$, then $T_{\mathbf{S}}(\mathcal{G})$ is uniform over $\mathsf{G}_{N,d}$.

Proof. We decompose the enlarged probability space according to the ℓ-neighborhood of 1 as

(3.26) $$\tilde{\mathsf{G}}_{N,d} = \bigcup_{\mathcal{T}} \tilde{\mathsf{G}}_{N,d}(\mathcal{T}), \quad \text{where } \tilde{\mathsf{G}}_{N,d}(\mathcal{T}) = \iota(\mathsf{G}_{N,d}(\mathcal{T})).$$

Notice that, given any $\mathcal{T}$, the size of the set $\mathsf{S}(\mathcal{G})$ is (by construction) independent of the graph $\mathcal{G} \in \mathsf{G}_{N,d}(\mathcal{T})$. Therefore, given any $\mathcal{T}$, the restricted measure $\tilde{\mathbb{P}}|\tilde{\mathsf{G}}_{N,d}(\mathcal{T})$ is uniform, i.e., proportional to the counting measure on the finite set $\tilde{\mathsf{G}}_{N,d}(\mathcal{T})$. Since, by Proposition 3.7, the map $\tilde{T}$ is an involution on $\tilde{\mathsf{G}}_{N,d}(\mathcal{T})$, it is in particular a bijection and as such preserves the uniform measure $\tilde{\mathbb{P}}|\tilde{\mathsf{G}}_{N,d}(\mathcal{T})$. Since $\tilde{T}$ acts diagonally in the decomposition (3.26), this implies that the map $\tilde{T}$ preserves the measure $\tilde{\mathbb{P}}$. Since $\mathbb{P} = \tilde{\mathbb{P}} \circ \pi^{-1}$ and $T = \pi \circ \tilde{T}$, it immediately follows that also T is measure preserving:

$$\tilde{\mathbb{P}} \circ T^{-1} = \tilde{\mathbb{P}} \circ \tilde{T}^{-1} \circ \pi^{-1} = \tilde{\mathbb{P}} \circ \pi^{-1} = \mathbb{P},$$

as claimed. $\qquad\square$

The following general proposition, which makes use of the involution property of $\tilde{T}$, is central to our approach. The idea of its proof is illustrated in Figure 6.

Proposition 3.9. *Given events $\Omega \subset \Omega^+ \subset \bar{\Omega} \subset \mathsf{G}_{N,d}$ and $\Omega' \subset \bar{\Omega}$, assume*

    (i) $\mathbb{P}(\mathsf{G}_{N,d} \setminus \bar{\Omega}) \leqslant q_0$,

    (ii) $\mathbb{P}_{\mathcal{G}}(T_{\mathbf{S}}(\mathcal{G}) \in \bar{\Omega} \setminus \Omega^+) \leqslant q_1$ for all $\mathcal{G} \in \Omega$, and

    (iii) $\mathbb{P}_{\mathcal{G}}(T_{\mathbf{S}}(\mathcal{G}) \in \bar{\Omega} \setminus \Omega') \leqslant q_2$ for all $\mathcal{G} \in \Omega^+$.

Then $\mathbb{P}(\Omega \setminus (\Omega \cap \Omega')) \leqslant q_0 + q_1 + q_2$.

Roughly, the proposition shows that if, for most graphs $\mathcal{G} \in \mathsf{G}_{N,d}$, an event holds for the switched graph $T_{\mathbf{S}}(\mathcal{G})$ with high probability under the randomness of the switching $\mathbf{S}$, then it also holds with high probability on $\mathsf{G}_{N,d}$. This enables us to condition on a (good) graph $\mathcal{G}$ for much of the paper, and then only use with the randomness of $\mathbf{S}$ which has a simple probabilistic structure.
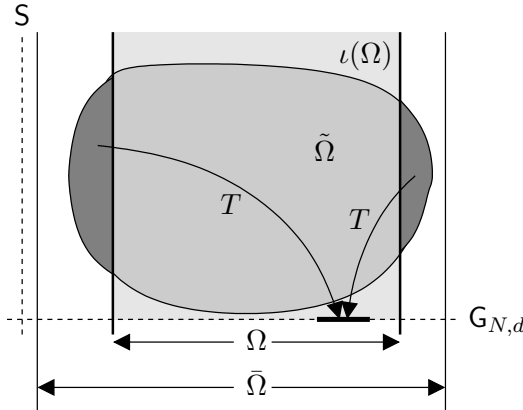
FIGURE 6. The figure illustrates the idea of Proposition 3.9. The horizontal axis represents the set of graphs $\mathsf{G}_{N,d}$, and the vertical direction the fibres of possible switchings. In particular, the sets Ω, Ω', Ω^+, $\bar{\Omega}$ are represented on the horizontal axis. The area in medium and dark grey represents $\tilde{\Omega} = T^{-1}(\Omega)$. The sets Ω' and Ω^+ and their preimages can be illustrated analogously, but for simplicity we assume for the figure that $\Omega = \Omega^+$. The lightly shaded area bounded by the vertical bars is $\iota(\Omega)$. In (3.28), we devide $\tilde{\Omega} \setminus \tilde{\Omega}'$ into the part contained in $\iota(\Omega^+)$ (the second term) and the part outside of $\iota(\Omega^+)$ (the first term). The part inside $\iota(\Omega^+)$ is small because of assumption (iii). To bound the part outside $\iota(\Omega^+)$, we use that $\tilde{T}$ is an involution. This implies that the image under $\tilde{T}$ of the area in dark gray is contained in $\iota(\Omega)$ (thus its projection to the horizontal axis lies in Ω as shown in the figure), and not intersecting $\tilde{\Omega}^+$. Its contribution is small by assumption (ii), which implies that $\iota(\Omega)$ contains most of $\tilde{\Omega}^+$.

More specifically, in our application of the proposition, the set $\bar{\Omega}$ is a large set of regular graphs obeying rough a priori estimates (there are only few cycles), the set Ω is a set of graphs for which the Green's function obeys good estimates, and the set Ω' is a sets of graphs on which the Green's function obeys better estimates (near a given vertex). The proposition states that if with respect to the resampling most graphs obey the better estimates, then these estimates also hold on the original probability space with high probability. The proposition will be applied in Section 3.11.

Proof. We define $\tilde{\Omega} = T^{-1}(\Omega)$, $\tilde{\Omega}' = T^{-1}(\Omega')$ and $\tilde{\Omega}^+ = T^{-1}(\Omega^+)$, and abbreviate $A \setminus B = A \setminus (A \cap B)$ for any sets A, B. Since

$$(3.27) \qquad T^{-1}(\Omega \setminus \Omega') = T^{-1}(\Omega) \setminus T^{-1}(\Omega') = \tilde{\Omega} \setminus \tilde{\Omega}',$$

and since T is measure preserving, and since $\tilde{T}$ is a measure preserving involution, we have

$$\mathbb{P}(\Omega \setminus \Omega') = \tilde{\mathbb{P}}(\tilde{\Omega} \setminus \tilde{\Omega}') = \tilde{\mathbb{P}}(\tilde{\Omega} \setminus (\tilde{\Omega}' \cup \iota(\Omega^+))) + \tilde{\mathbb{P}}((\iota(\Omega^+) \cap \tilde{\Omega}) \setminus \tilde{\Omega}')$$

$$\leqslant \tilde{\mathbb{P}}(\tilde{\Omega} \setminus (\tilde{\Omega}' \cup \iota(\Omega^+))) + \tilde{\mathbb{P}}((\iota(\Omega^+) \cap T^{-1}(\bar{\Omega})) \setminus \tilde{\Omega}')$$

$$(3.28) \qquad = \tilde{\mathbb{P}}(\hat{\Omega}) + \tilde{\mathbb{P}}((\iota(\Omega^+) \cap T^{-1}(\bar{\Omega})) \setminus \tilde{\Omega}'),$$

where $\hat{\Omega} = \tilde{T}(\tilde{\Omega} \setminus (\tilde{\Omega}' \cup \iota(\Omega^+)))$. To bound the probability of $\hat{\Omega}$, we make the following observations. First, $\hat{\Omega} \subset \tilde{T}(\tilde{\Omega}) \subset \iota(\Omega)$. Second, any element $(\hat{\mathcal{G}}, \hat{\mathbf{S}}) \in \hat{\Omega}$ can be written as $\hat{\mathcal{G}} = T(\mathcal{G}, \mathbf{S})$ for some $\mathcal{G} \notin \Omega^+$ and $\mathbf{S} \in \mathsf{S}(\mathcal{G})$. Since $\tilde{T}$ is an involution, this $(\mathcal{G}, \mathbf{S})$ must in fact be given by $(\mathcal{G}, \mathbf{S}) = \tilde{T}(\hat{\mathcal{G}}, \hat{\mathbf{S}})$. Together this implies that $(\hat{\mathcal{G}}, \hat{\mathbf{S}}) \notin \tilde{\Omega}^+$, and thus that $\hat{\Omega}$ has no intersection with $\tilde{\Omega}^+$. As a result,

$$\mathbb{P}(\Omega \setminus \Omega') \leqslant \tilde{\mathbb{P}}(\iota(\Omega) \setminus \tilde{\Omega}^+) + \tilde{\mathbb{P}}((\iota(\Omega^+) \cap T^{-1}(\bar{\Omega})) \setminus \tilde{\Omega}')$$

$$= \tilde{\mathbb{P}}(\iota(\Omega) \setminus T^{-1}(\bar{\Omega})) + \tilde{\mathbb{P}}((\iota(\Omega) \cap T^{-1}(\bar{\Omega})) \setminus \tilde{\Omega}^+) + \tilde{\mathbb{P}}((\iota(\Omega^+) \cap T^{-1}(\bar{\Omega})) \setminus \tilde{\Omega}')$$

$$\leqslant \tilde{\mathbb{P}}(T^{-1}(\mathsf{G}_{N,d}) \setminus T^{-1}(\bar{\Omega})) + \tilde{\mathbb{P}}((\iota(\Omega) \cap T^{-1}(\bar{\Omega})) \setminus \tilde{\Omega}^+) + \tilde{\mathbb{P}}((\iota(\Omega^+) \cap T^{-1}(\bar{\Omega})) \setminus \tilde{\Omega}')$$

$$\leqslant q_0 + q_1 + q_2,$$

where the second inequality follows since $\mathsf{G}_{N,d} \supset \Omega$ and the last inequality follows from the assumptions (i)–(iii). $\qquad\square$

3.3.3. *Boundary of neighborhood* $\mathbb{T}$.

Proposition 3.10. *Let $\mathcal{G}$ be a d-regular graph on $[\![N]\!]$, assume that $\mathcal{B}_R(1, \mathcal{G})$ has excess at most ω, and that $\ell \ll R$. Then the following hold.*

- *After removing $\mathbb{T}$, most boundary vertices of $\mathcal{T}$ are isolated from the other boundary vertices:*

(3.29) $$|\{p \in [\![1, \mu]\!] : \exists q \in [\![1, \mu]\!] \setminus \{p\}, \operatorname{dist}_{\mathcal{G}^{(\mathbb{T})}}(a_p, a_q) \leqslant R/2\}| \leqslant 2\omega.$$

- *After removing $\mathbb{T}$, any vertex $x \in [\![N]\!] \setminus \mathbb{T}$ can only be close to few boundary vertices of $\mathbb{T}$:*

(3.30) $$|\{p \in [\![1, \mu]\!] : \operatorname{dist}_{\mathcal{G}^{(\mathbb{T})}}(x, a_p) \leqslant R/2\}| \leqslant \omega + 1,$$

(3.31) $$|\{v \in \mathbb{T}_\ell : \operatorname{dist}_{\mathcal{G} \setminus \mathcal{T}}(x, v) \leqslant R/2\}| \leqslant \omega + 1.$$

Notice that the graph $\mathcal{G} \setminus \mathcal{T}$ is slightly larger than $\mathcal{G}^{(\mathbb{T})}$ because the edges between the vertices $\mathbb{T}_\ell$ and $[\![N]\!] \setminus \mathbb{T}$ are not removed.

Bound on deficit functions. Finally, we have the following deterministic bound on the deficit functions for the connected components of the subgraph obtained from $\mathcal{B}_R(1, \mathcal{G})$ by removing a set of vertices $\mathbb{U}$.

Proposition 3.11. *Let $\mathcal{G}$ be a d-regular graph on $[\![N]\!]$, and assume that $\mathcal{B} := \mathcal{B}_R(1, \mathcal{G})$ has excess at most ω. Then the following hold.*

- *Let $\mathcal{A}$ be the annulus obtained by removing $\mathbb{T}$ from $\mathcal{B}$. Then the sum of the deficit function over any connected component of $\mathcal{A}$ satisfies $\sum g(v) \leqslant \omega + 1$.*
- *Given $\mathbb{U} \subset \mathbb{B}_\ell(1, \mathcal{G})$, let $\mathcal{B}^{(\mathbb{U})}$ be the subgraph given by removing the vertices $\mathbb{U}$ from $\mathcal{B}$. Then the sum of the deficit function over any connected component of $\mathcal{B}^{(\mathbb{U})}$ satisfies $\sum g(v) \leqslant \omega + |\mathbb{U}|$.*

For the above statements, recall that we view $\mathcal{A}$ and $\mathcal{B}^{(\mathbb{U})}$ as subgraphs of $\mathcal{B}$ (which has zero deficit function) and that their deficit functions are given by our conventions in Section 2.3.

In the remainder of this section, we prove Propositions 3.10 and 3.11.
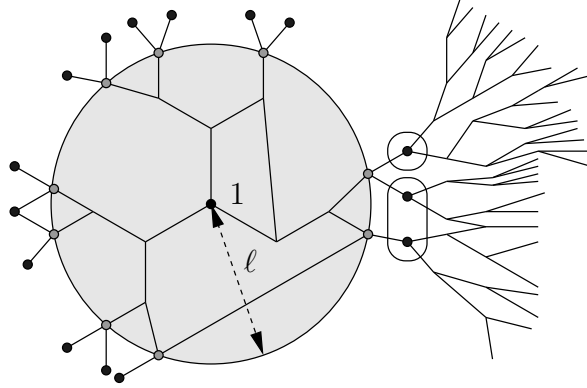
FIGURE 7. The two vertices a_i which are encircled together are close in the sense that they are in the same connected component of the annulus $\mathcal{A}$. Proposition 3.10 shows that, since $\mathcal{B}$ has excess at most ω, this happens for at most 2ω of the a_i. In addition, it shows that any vertex x outside $\mathbb{T}$ can only be close to at most $\omega + 1$ of the a_i.

3.3.4. *Proof of Proposition 3.10.* Abbreviate $\mathcal{B} = \mathcal{B}_R(1, \mathcal{G})$. By assumption the ball $\mathcal{B}$ has excess at most ω. Let $\mathcal{A}$ be the annulus obtained by removing $\mathbb{T}$ from $\mathcal{B}$. We partition $[\![1, \mu]\!]$ into sets $\{A_1, A_2, A_3, \dots\}$, such that i and j are in the same set A_k if and only if a_i and a_j are in the same connected component of $\mathcal{A}$. We label the sets A_k such that $|A_1| \geqslant |A_2| \geqslant \dots \geqslant |A_\alpha| > 1 = |A_{\alpha+1}| = \cdots$ and let i_j be a labeling such that $A_1 \cup \cdots \cup A_\alpha = \{i_1, i_2, \dots, i_\beta\}$.

Lemma 3.12.

(3.32)
$$\alpha \leqslant \omega, \quad \beta \leqslant 2\omega, \quad |A_j| \leqslant \omega + 1 \quad \text{for all } j.$$

Proof. For any finite graph $\mathcal{G}$, we set

(3.33)
$$\chi(\mathcal{G}) := \#\text{connected components}(\mathcal{G}) - \text{excess}(\mathcal{G}) = \#\text{vertices}(\mathcal{G}) - \#\text{edges}(\mathcal{G}),$$

where the second equality follows from the definition (2.5) of $\text{excess}(\mathcal{G})$. In particular, for any $e \in \mathcal{G}$, we have $\chi(\mathcal{G} \setminus e) = \chi(\mathcal{G}) + 1$.

As a ball, $\mathcal{B}$ has by definition exactly one connected component, and by assumption it has excess at most ω. Thus $\chi(\mathcal{B}) \geqslant 1 - \omega$. We recall that e_i is the edge on the boundary of $\mathcal{T}$ containing a_i. Thus the graph $\mathcal{B} \setminus \{e_{i_1}, \ldots, e_{i_\beta}\}$ has at most $\alpha + 1$ connected components: the component containing the vertex 1 and the components containing the vertices a_i with $i \in A_j$ for some $j \in [\![1, \alpha]\!]$. (Notice for $i \in A_j$ with $j > \alpha$, we did not remove the edge e_i. Therefore a_i is still connected to 1.) Thus $\chi(\mathcal{B} \setminus \{e_{i_1}, \ldots, e_{i_\beta}\}) \leqslant \alpha + 1$. It follows that

$$1 + \alpha \geqslant \chi(\mathcal{B} \setminus \{e_{i_1}, \ldots, e_{i_\beta}\}) = \chi(\mathcal{B}) + \beta \geqslant 1 - \omega + \beta,$$

and thus $\beta \leqslant \alpha + \omega$. Since, by definition, we have $\beta = \sum_{i=1}^{\alpha} |A_i| \geqslant 2\alpha$, the first two inequalities in (3.32) follow. The third inequality is trivial for $i > \alpha$, and for $i \leqslant \alpha$, we have

$$\omega + \alpha \geqslant \beta = \sum_{j=1}^{\alpha} |A_j| \geqslant |A_i| + 2(\alpha - 1),$$

which implies that $|A_i| \leqslant \omega - \alpha + 2 \leqslant \omega + 1$ as claimed. □

Proof of (3.29). By definition, any i, j such that $\mathrm{dist}_{\mathcal{G}^{(\mathbb{T})}}(a_i, a_j) \leqslant R/2$ belong to the same connected component of $\mathcal{A}$. (Indeed, a_i is at distance $\ell + 1$ from the vertex 1 and $R \gg \ell$, and thus $\mathcal{B}_{R/2}(a_i, \mathcal{G}) \subset \mathcal{B}$ for any $i \in [\![1, \mu]\!]$.) In particular, if the set A_i containing i has size 1, then for any $j \in [\![1, \mu]\!] \setminus \{i\}$, we have $\mathrm{dist}_{\mathcal{G}^{(\mathbb{T})}}(a_i, a_j) > R/2$. Recalling that $\beta \leqslant 2\omega$ is the number of i for which the set A_i containing it has size greater than 1, the claim (3.29) follows from (3.32). □

Proof of (3.30). The claim is trivial if $x \notin \mathcal{B}$, since we then have $\mathrm{dist}_{\mathcal{G}^{(\mathbb{T})}}(x, \{a_1, a_2, \ldots, a_\mu\}) \geqslant R - \ell > R/2$ by definition. Thus assume that $x \in \mathcal{B}$. Let A_j be such that x and the vertices a_i with $i \in A_j$ are in the same connected component of $\mathcal{A}$. We first show that those vertices a_p with $p \in A_k$ where $k \neq j$ do not contribute to (3.30). Indeed, then x and a_p are in the different connected components of $\mathcal{A}$. But since $\mathcal{B}_{R/2}(a_p, \mathcal{G}) \subset \mathcal{B}$, it then follows that $\mathrm{dist}_{\mathcal{G}^{(\mathbb{T})}}(x, a_p) > R/2$. Therefore

$|\{p \in [\![1, \mu]\!] : \mathrm{dist}_{\mathcal{G}^{(\mathbb{T})}}(x, a_p) \leqslant R/2\}| \leqslant |A_j|$, and the claim (3.30) follows from the third inequality in (3.32). $\qquad\square$

Proof of (3.31). By the same proof, (3.30) also holds with ℓ replaced by $\ell - 1$, i.e., with $\mathbb{T} = \mathbb{B}_\ell(1, \mathcal{G})$ replaced by $\mathbb{B}_{\ell-1}(1, \mathcal{G})$, including in Lemma 3.12. This gives

$$|\{v \in \mathbb{T}_\ell : \mathrm{dist}_{\mathcal{G}^{(\mathbb{B}_{\ell-1}(1,\mathcal{G}))}}(x, v) \leqslant R/2\}| \leqslant \omega + 1.$$

Then claim then follows since $\mathcal{G} \setminus \mathcal{T} \subset \mathcal{G}^{(\mathbb{B}_{\ell-1}(1,\mathcal{G}))}$. $\qquad\square$

3.3.5. *Proof of Proposition 3.11.*

Proof of Proposition 3.11. For the first statement, viewing the annulus $\mathcal{A}$ as a subgraph of $\mathcal{G}^{(\mathbb{T})}$, the bound (3.32) immediately implies that the sum of deficit function over any connected component of $\mathcal{A}$ satisfies $\sum g(v) \leqslant \max_j |A_j| \leqslant \omega + 1$.

For the second statement, let $k = |\mathbb{U}|$ and write $\mathbb{U} = \{u_1, u_2, \ldots, u_k\}$. Let $\mathbb{X} \subset [\![N]\!]$ be the set of vertices of any given connected component of $\mathcal{A}$. Define

$$B_i := \mathbb{X} \cap \partial u_i = \left\{v_1^i, v_2^i, \ldots, v_{|B_i|}^i\right\}, \quad i = 1, 2, \ldots, k,$$

where ∂u is the set of neighbors of the vertex u in $\mathcal{G}$. Notice that $g(v) = 0$ unless $v \in B_1 \cup \cdots \cup B_k$. Thus

(3.34) $$\sum_{v \in \mathbb{X}} g(v) \leqslant \sum_{i=1}^k \sum_{v \in B_k} g(v) \leqslant \sum_{i=1}^k |B_i| \leqslant |\mathbb{U}| + \sum_{i=1}^k (|B_i| - 1),$$

so that the claim follows from

(3.35) $$\sum_{i=1}^k (|B_i| - 1) \leqslant \omega.$$

To prove (3.35), we consider the graph

$$\mathcal{H} = \mathcal{B} \setminus \bigcup_{i=1}^k \{\{u_i, v_1^i\}, \{u_i, v_2^i\}, \cdots, \{u_i, v_{|B_i|-1}^i\}\}.$$

Note that $\mathcal{H}$ is obtained from $\mathcal{B}$ by removing exactly $\sum_{i=1}^{k}(|B_i| - 1)$ edges and that $\mathcal{H}$ is connected. Since by assumption $\mathcal{B}$ has excess at most ω, after removing any $\omega + 1$ edges, it cannot be connected. This implies (3.35). $\qquad \square$

3.3.6. *Estimates for local resampling.* In the following, we give some basic estimates for the local resampling. In particular, we show that, with high probability, most edges are switchable.

Proposition 3.13. *Let $\delta > 0$.*

(i) *For any $x \in [\![N]\!] \setminus \mathbb{T}$,*

$$(3.36) \qquad \mathbb{P}_{\mathcal{G}}(b_i = x) = \mathbb{P}(c_i = x) \leqslant \frac{2}{N}.$$

(ii) *For any positive integer ω, we have*

$$(3.37) \qquad \mathbb{P}_{\mathcal{G}}(|W_{\mathbf{S}}| > \mu - 3\omega) = 1 - \mathrm{o}(N^{-\omega+\delta}).$$

Proof. To prove (i), we recall that, for any i, the oriented edge (b_i, c_i) is uniformly chosen from the oriented edges of $\mathcal{G}^{(\mathbb{T})}$. By definition of $\mathbb{T}$, there are at least $Nd/2 - (d + d(d-1) + \cdots d(d-1)^{\ell})$ edges in $\mathcal{G}^{(\mathbb{T})}$, and since for any vertex $x \in \mathcal{G}^{(\mathbb{T})}$, the degree obeys $\deg_{\mathcal{G}^{(\mathbb{T})}}(x) \leqslant d$,

$$\mathbb{P}_{\mathcal{G}}(b_i = x) = \mathbb{P}_{\mathcal{G}}(c_i = x) \leqslant \frac{d}{Nd - 2(d + d(d-1) + \cdots d(d-1)^{\ell})} \leqslant \frac{2}{N}.$$

To prove (ii), we need to analyze the events $I_i J_i = 0$ more carefully. We define the disjoint sets

$$A_0 = \{i \in [\![1, \mu]\!] : I_i = 0\},$$

$$A_1 = \{i \in [\![1, \mu]\!] \setminus A_0 : |\{b_i, c_i\} \cap (\cup_{j \neq i}[e_j])| \geqslant 1\},$$

$$A_2 = \{i \in [\![1, \mu]\!] \setminus A_0 \cup A_1 : |\{b_i, c_i\} \cap (\cup_{j \neq i}\{b_j, c_j\})| \geqslant 1\},$$

$$A_3 = \{i \in [\![1, \mu]\!] \setminus A_0 \cup A_1 \cup A_2 : \exists\, j \text{ such that } l_i = l_j \text{ and } |[e_i] \cap \{b_j, c_j\}| \geqslant 1\},$$

and claim that

$$(3.38) \qquad [\![1, \mu]\!] \setminus W_{\mathbf{S}} = \{i \in [\![1, \mu]\!] : I_i J_i = 0\} \subset A_0 \cup A_1 \cup A_2 \cup A_3.$$

Indeed, if $i \in [\![1, \mu]\!] \setminus W_{\mathbf{S}}$, then $I_i = 0$ or $J_i = 0$. Clearly, if $I_i = 0$ then $i \in A_0 \subset A_0 \cup A_1 \cup A_2 \cup A_3$. On the other hand, if $J_i = 0$, there exists some index $j \in [\![1, \mu]\!] \setminus \{i\}$ such that $|[S_i] \cap [S_j]| > 1$, and there are two possibilities (recall that $e_i = \{l_i, a_i\}$ and $e_j = \{l_j, a_j\}$):

(i) $l_i \neq l_j$. Then either $|\{b_i, c_i\} \cap \{b_j, c_j\}| \geqslant 1$; or $|\{b_i, c_i\} \cap [e_j]| \geqslant 1$ and $|[e_i] \cap \{b_j, c_j\}| \geqslant 1$.

(ii) $l_i = l_j$. Then either $|\{b_i, c_i\} \cap \{b_j, c_j\}| \geqslant 1$; or $|\{b_i, c_i\} \cap [e_j]| \geqslant 1$; or $|[e_i] \cap \{b_j, c_j\}| \geqslant 1$.

Either way, $J_i = 0$ implies $i \in A_1 \cup A_2 \cup A_3$, and (3.38) holds. To bound the number of elements on the right-hand side of (3.38), we first note that $|A_3| \leqslant 2|A_0 \cup A_1|$. In fact if $i \in A_3$, then there exists some j such that $|[e_i] \cap \{b_j, c_j\}| \geqslant 1$, and thus $j \in A_0 \cup A_1$. Since any $\{b_j, c_j\}$ can intersect at most two edges e_i with $l_i = l_j$,

$$|A_3| \leqslant \sum_{j \in A_0 \cup A_1} |\{i \in [\![1, \mu]\!] : l_i = l_j \text{ and } |\{b_j, c_j\} \cap [e_i]| \geqslant 1\}| \leqslant 2|A_0 \cup A_1|.$$

Therefore, it follows that

(3.39)

$$|[\![1,\mu]\!] \setminus W_{\mathbf{S}}| = |\{i \in [\![1,\mu]\!] : I_i J_i = 0\}| \leqslant 3|A_0 \cup A_1| + |A_2| \leqslant 3|A_0| + 3|A_1| + |A_2|.$$

We will show that

(3.40)
$$\mathbb{P}(|A_0| + |A_1| + \tfrac{1}{2}|A_2| \geqslant \omega) = \mathrm{o}(N^{-\omega+\delta}),$$

which implies the claim since

$$\mathbb{P}(|[\![1,\mu]\!] \setminus W_{\mathbf{S}}| \geqslant 3\omega) \leqslant \mathbb{P}(3|A_0| + 3|A_1| + |A_2| \geqslant 3\omega)$$

$$\leqslant \mathbb{P}(|A_0| + |A_1| + \tfrac{1}{2}|A_2| \geqslant \omega) = \mathrm{o}(N^{-\omega+\delta}).$$

To prove (3.40), first notice that there is a subset $A_2' \subset A_2$ with $|A_2'| \geqslant |A_2|/2$ such that $i \in A_2'$ implies $|\{b_i, c_i\} \cap (\cup_{j \notin A_2'} \{b_j, c_j\})| \geqslant 1$. Hence, if $|A_0| + |A_1| + |A_2|/2 \geqslant \omega$, then there exist disjoint index sets $\tilde{A}_0 \subset A_0, \tilde{A}_1 \subset A_1, \tilde{A}_2' = A_2'$ such that $|\tilde{A}_0| + |\tilde{A}_1| + |\tilde{A}_2'| = \omega$ and

(3.41)
$$\begin{cases} \forall i \in \tilde{A}_0, & I_i = 0, \\[2mm] \forall i \in \tilde{A}_1, & |\{b_i, c_i\} \cap (\cup_{j \neq i}[e_j])| \geqslant 1, \\[2mm] \forall i \in \tilde{A}_2', & |\{b_i, c_i\} \cap (\cup_{j \notin A_2'} \{b_j, c_j\})| \geqslant 1. \end{cases}$$

The condition $I_i = 0$ is equivalent to $\mathrm{dist}_{\mathcal{G}}(\{a_i, l_i\}, \{b_i, c_i\}) \leqslant 1$. Therefore, by (3.36),

(3.42) $\qquad \mathbb{P}_{\mathcal{G}}(I_i = 0) \leqslant \mathbb{P}_{\mathcal{G}}(\mathrm{dist}_{\mathcal{G}}(\{a_i, l_i\}, b_i) \leqslant 1) + \mathbb{P}_{\mathcal{G}}(\mathrm{dist}_{\mathcal{G}}(\{a_i, l_i\}, c_i) \leqslant 1)$

(3.43) $\qquad\qquad\qquad \leqslant \dfrac{4}{N} \#\{x \in \mathcal{G}^{(\mathbb{T})} : \mathrm{dist}_{\mathcal{G}}(\{a_i, l_i\}, x) \leqslant 1\} \leqslant \dfrac{8d}{N}.$

Similarly, since $|\cup_j [e_j]| \leqslant 2\mu$,

(3.44) $\qquad\qquad\qquad \mathbb{P}_{\mathcal{G}}(|\{b_i, c_i\} \cap (\cup_{j \neq i}[e_j])| \geqslant 1) \leqslant \dfrac{8\mu}{N},$

and, for any $i \in \tilde{A}_2'$, we have

$$(3.45) \qquad \mathbb{P}_{\mathcal{G}}\left(|\{b_i, c_i\} \cap (\cup_{j \notin \tilde{A}_1'}\{b_j, c_j\}| \geq 1 \big| \vec{S}_j, j \notin \tilde{A}_2'\right) \leq \frac{8\mu}{N}.$$

Finally, there are at most $(3\mu)^\omega$ disjoint sets $\tilde{A}_0, \tilde{A}_1, \tilde{A}_2' \subset [\![1, \mu]\!]$ such that $|\tilde{A}_0| + |\tilde{A}_1| + |\tilde{A}_2'| = \omega$, and therefore

$$\mathbb{P}_{\mathcal{G}}\left(|A_0| + |A_1| + \tfrac{1}{2}|A_2| \geq \omega\right) \leq (3\mu)^\omega \max_{\tilde{A}_0, \tilde{A}_1, \tilde{A}_2'} \mathbb{P}_{\mathcal{G}}\left(\text{the sets } \tilde{A}_0, \tilde{A}_1, \tilde{A}_2' \text{ satisfy } (3.41)\right)$$

$$= (3\mu)^\omega \max_{\tilde{A}_0, \tilde{A}_1, \tilde{A}_2'} \prod_{i \in \tilde{A}_0} \mathbb{P}_{\mathcal{G}}(I_i = 0) \prod_{i \in \tilde{A}_1} \mathbb{P}_{\mathcal{G}}(|\{b_i, c_i\} \cap (\cup_{j \neq i}[e_j])| \geq 1)$$

$$\prod_{i \in \tilde{A}_2'} \mathbb{P}_{\mathcal{G}}\left(|\{b_i, c_i\} \cap (\cup_{j \notin \tilde{A}_2'}\{b_j, c_j\}| \geq 1 \big| \vec{S}_j, j \notin \tilde{A}_2'\right)$$

$$\leq (3\mu)^\omega \max_{\tilde{A}_0, \tilde{A}_1, \tilde{A}_2'} \prod_{i \in \tilde{A}_0} \frac{8d}{N} \prod_{i \in \tilde{A}_1} \frac{8\mu}{N} \prod_{i \in \tilde{A}_2'} \frac{8\mu}{N} = o(N^{-\omega+\delta}),$$

where the maxima are over all disjoint sets $\tilde{A}_0, \tilde{A}_1, \tilde{A}_2' \subset [\![1, \mu]\!]$ such that $|\tilde{A}_0| + |\tilde{A}_1| + |\tilde{A}_2'| = \omega$, and where we used that the probability factorizes since the sets $\tilde{A}_0, \tilde{A}_1, \tilde{A}_2'$ are disjoint. $\qquad \square$

Remark 3.14. *Throughout Sections 3.3.7–3.10, we fix a d-regular graph $\mathcal{G} \in \mathsf{G}_{N,d}$ on the vertex set $[\![N]\!]$, and abbreviate its ℓ-neighborhood of 1 by*

$$(3.46) \qquad \mathbb{T} = \mathbb{B}_\ell(1, \mathcal{G}), \quad \mathcal{T} = \mathcal{B}_\ell(1, \mathcal{G}).$$

We also write

$$(3.47) \qquad \mathbb{T}_i = \{v \in \mathcal{G} : \mathrm{dist}_{\mathcal{G}}(1, v) = i\},$$

for the set of vertices at distance i from 1.

Further, we enumerate the boundary edges $\partial_E \mathcal{T}$ as $\{l_i, a_i\}$ for $i \in [\![1, \mu]\!]$, where $l_i \in \mathbb{T}$ and $a_i \in [\![N]\!] \setminus \mathbb{T}$. We denote the resampling data by $\mathbf{S} = (\vec{S}_1, \vec{S}_2, \ldots, \vec{S}_\mu)$, where $\vec{S}_i = \{(l_i, a_i), (b_i, c_i)\}$ for $i \in [\![1, \mu]\!]$, and $(b_1, c_1), \ldots, (b_\mu, c_\mu)$ are chosen independently
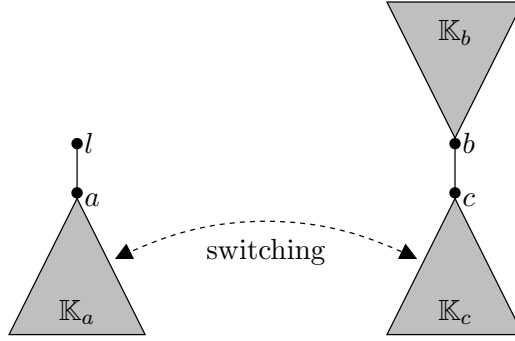
FIGURE 8. For the vertices $x \in \{a_i, b_i, c_i\}$ participating in the switching, we denote by $\mathbb{K}_x$ their radius-$R/4$ neighborhoods in the unswitched graph, with b_i and c_i disconnected and the set $\mathbb{T}$ removed. The typical $\mathbb{K}_x$ are disjoint from the other $\mathbb{K}_x$, and in the typical case, the sets $\mathbb{K}_a$ and $\mathbb{K}_c$ exchange their roles under switching.

and uniformly among oriented edges from the graph $\mathcal{G}^{(\mathbb{T})}$. We denote by $\mathsf{S}(\mathcal{G})$ the set of all possible switching data, so that $\mathbf{S}$ is uniformly distributed on S. Given switching data $\mathbf{S}$, we denote the set of admissible switchings by $W_{\mathbf{S}}$. Without loss of generality, we will assume for notational convenience that $W_{\mathbf{S}} = \{1, 2, 3, \ldots, \nu\}$ where $|W_{\mathbf{S}}| = \nu \leqslant \mu$.

To study the change of graphs before and after local resampling, we define the following graphs (which need not be regular).

- $\mathcal{G}$ is the original unswitched graph;
- $\mathcal{G}^{(\mathbb{T})}$ is the unswitched graph with vertices $\mathbb{T}$ removed;
- $\hat{\mathcal{G}}^{(\mathbb{T})}$ is the intermediate graph obtained from $\mathcal{G}^{(\mathbb{T})}$ by removing the edges $\{b_i, c_i\}$ with $i \in W_{\mathbf{S}}$;
- $\tilde{\mathcal{G}}^{(\mathbb{T})}$ is the switched graph obtained from $\hat{\mathcal{G}}^{(\mathbb{T})}$ by adding the edges $\{a_i, b_i\}$ with $i \in W_{\mathbf{S}}$; and
- $\tilde{\mathcal{G}}$ is the switched graph $T_{\mathbf{S}}(\mathcal{G})$ (including vertices $\mathbb{T}$).

Following the conventions of Section 2.3, the deficit functions of these graphs are given by $d - \deg$, where $\deg$ is the degree function of the graph considered, and we abbreviate their Green's functions by G, $G^{(\mathbb{T})}$, $\hat{G}^{(\mathbb{T})}$, $\tilde{G}^{(\mathbb{T})}$, and $\tilde{G}$ respectively.

3.3.7. *Graph distance between switched vertices.* This section provides estimates on the distances between the vertices participating in the switching, in the graph with vertices $\mathbb{T}$ removed (before and after switching).

It can be helpful to think about these estimates in terms of the sets $\mathbb{K}_x \subset [\![N]\!] \setminus \mathbb{T}$ defined by

$$(3.48) \quad \mathbb{K}_{a_i} = \mathbb{B}_{R/4}(a_i, \mathcal{G}^{(\mathbb{T})}), \quad \mathbb{K}_{x_i} = \mathbb{B}_{R/4}(x_i, \mathcal{G}^{(\mathbb{T})} \setminus \{\{b_i, c_i\}\}), \quad \text{where } x_i \in \{b_i, c_i\},$$

and illustrated in Figure 8. In (3.29), (3.30), it is shown that

- (3.29) except for at most 2ω many, the $\mathbb{K}_a$ does not intersect the other $\mathbb{K}_a$.
- (3.30) any $x \in [\![N]\!] \setminus \mathbb{T}$ is in at most $\omega + 1$ many of the sets $\mathbb{K}_a$.

Roughly speaking, in this section it is shown that, for any graph $\mathcal{G} \in \bar{\Omega}$, the following estimates hold with high probability under $\mathbb{P}_\mathcal{G}$:

- (3.49) any $x \in [\![N]\!] \setminus \mathbb{T}$ is in at most ω of the sets $\mathbb{K}_b$;
- (3.50) any $\mathbb{K}_a$ intersects at most ω of the $\mathbb{K}_b$;
- (3.51) any $\mathbb{K}_b$ intersects at most 2ω of the other $\mathbb{K}_a, \mathbb{K}_b$;
- (3.52) except for at most ω many, the $\mathbb{K}_b$ are trees.

By symmetry, the same statements hold with b replaced by c. More precisely, in the remainder of this section, we show that the estimates stated in the following propositions hold.

Proposition 3.15. *For any graph $\mathcal{G} \in \bar{\Omega}$ (as in Section 3.1.2), the following holds with $\mathbb{P}_\mathcal{G}$-probability at least $1 - \mathrm{o}(N^{-\omega+\delta})$:*

- *Any vertex $x \in [\![N]\!] \setminus \mathbb{T}$ is far away from most vertices in $\{b_1, b_2, \ldots, b_\mu\}$:*

$$(3.49) \qquad\qquad |\{i \in [\![1, \mu]\!] : \mathrm{dist}_{\mathcal{G}^{(\mathbb{T})}}(x, b_i) \leqslant R/2\}| < \omega.$$

- *Most indices $i \in [\![1, \mu]\!]$ are good:*

(3.50)

$$|\mathsf{B}_a| < 3\omega, \quad with \ \mathsf{B}_a = \{i \in [\![1, \mu]\!] : \mathrm{dist}_{\mathcal{G}^{(\mathbb{T})}}(a_i, \{a_j, b_k : j \in [\![1, \mu]\!] \setminus \{i\}, k \in [\![1, \mu]\!]\}) \leqslant R/2\},$$

(3.51)

$$|\mathsf{B}_b| < 2\omega, \quad with \ \mathsf{B}_b = \{i \in [\![1, \mu]\!] : \mathrm{dist}_{\mathcal{G}^{(\mathbb{T})}}(b_i, \{a_j, b_k : j \in [\![1, \mu]\!], k \in [\![1, \mu]\!] \setminus \{i\}\}) \leqslant R/2\},$$

(3.52)

$$|\mathsf{B}_c| < \omega, \quad with \ \mathsf{B}_c = \{i \in [\![1, \mu]\!] : \mathcal{B}_R(c_i, \mathcal{G}^{(\mathbb{T})}) \ is \ not \ a \ tree \ \}.$$

Note that B_a is the set of indices i such that $\mathbb{K}_{a_i}$ is not disjoint from all sets other $\mathbb{K}_a$ and $\mathbb{K}_b$, and that B_b is the set of indices i such that $\mathbb{K}_{b_i}$ is not disjoint from all other sets $\mathbb{K}_b$ and $\mathbb{K}_a$.

We will show that the estimates (3.50) and (3.51) also imply the following estimates for the switched graph $\tilde{\mathcal{G}}^{(\mathbb{T})}$.

Proposition 3.16. *Assume* (3.50) *and* (3.51).

- *For any index $i \in [\![1, \mu]\!] \setminus (\mathsf{B}_a \cup \mathsf{B}_b)$,*

(3.53) $$\mathrm{dist}_{\tilde{\mathcal{G}}^{(\mathbb{T})}}(\{a_i, b_i\}, \{a_j, b_j : j \in [\![1, \mu]\!] \setminus \{i\}\}) > R/2.$$

- *For any vertex $x \in [\![N]\!] \setminus \mathbb{T}$,*

(3.54) $$|\{i \in [\![1, \mu]\!] : \mathrm{dist}_{\tilde{\mathcal{G}}^{(\mathbb{T})}}(x, \{a_i, b_i\}) \leqslant R/4\}| \leqslant 5\omega.$$

The remainder of this section is devoted to the proofs of Propositions 3.15–3.16.

3.3.8. *Proof of Proposition 3.15.* Recall that the oriented edges (b_i, c_i) are independent and distributed approximately uniformly, so that (3.36) holds. The claims essentially follow from this.

Proof of (3.49). In any graph with degree bounded by d, the number of vertices at distance at most $R/2$ from vertex x is bounded by $1+d+d(d-1)+\cdots+d(d-1)^{R/2-1} \leqslant 2(d-1)^{R/2}$. By (3.36) therefore

$$(3.55) \qquad\qquad \mathbb{P}_{\mathcal{G}}(\text{dist}_{\mathcal{G}^{(\mathbb{T})}}(x, b_i) \leqslant R/2) \leqslant \frac{4(d-1)^{R/2}}{N}.$$

Since the $b_1, \ldots, b_\mu$ are independent, it therefore follows that

$$\mathbb{P}\left(|\{i \in [\![1, \mu]\!] : \text{dist}_{\mathcal{G}^{(\mathbb{T})}}(x, b_i) \leqslant R/2\}| \geqslant \omega\right) \leqslant \binom{\mu}{\omega}\left(\frac{4(d-1)^{R/2}}{N}\right)^\omega \ll N^{-\omega+\delta},$$

where, in the last inequality, we used that $(4(d-1)^{R/2}\mu)^\omega \leqslant 2^{3\omega}(d-1)^{(R/2+\ell+1)\omega} \ll N^\delta$ by the choice of parameters in Section 3.1. $\qquad\square$

Proof of (3.50). Recall the annulus $\mathcal{A}$, and sets $A_1, A_2, \ldots$ from Lemma 3.12. By (3.32), $|A_1 \cup \cdots \cup A_\alpha| \leqslant 2\omega$, and for any $i \in A_{\alpha+1} \cup A_{\alpha+2} \cup \cdots$, a_i is at least distance R in $\mathcal{G}^{(\mathbb{T})}$ from other vertices a_j. It follows that

$$\mathbb{P}_{\mathcal{G}}\left(|\{i \in [\![1, \mu]\!] : \text{dist}_{\mathcal{G}^{(\mathbb{T})}}(a_i, \{a_j, b_k : j \in [\![1, \mu]\!] \setminus \{i\}, k \in [\![1, \mu]\!]\})\}| \leqslant R/2\} \geqslant 3\omega\right)$$

$$\leqslant \mathbb{P}_{\mathcal{G}}\left(|\{i \in A_{\alpha+1} \cup A_{\alpha+2} \cup \cdots : \text{dist}_{\mathcal{G}^{(\mathbb{T})}}(a_i, \{b_1, b_2, \ldots, b_\mu\})\}| \leqslant R/2\} \geqslant \omega\right).$$

By a union bound, the right-hand side is bounded by

$$\sum_{A', B'} \mathbb{P}_{\mathcal{G}}\left(\text{dist}_{\mathcal{G}^{(\mathbb{T})}}(a_{i_1}, b_{j_1}) \leqslant R/2, \ldots, \text{dist}_{\mathcal{G}^{(\mathbb{T})}}(a_{i_\omega}, b_{j_\omega}) \leqslant R/2\right),$$

where $A' = \{i_1, \ldots, i_\omega\}$, $B' = \{j_1, \ldots, j_\omega\}$, and the sum over A' runs through the subsets of $A_{\alpha+1} \cup A_{\alpha+2} \cup \cdots$ with $|A'| = \omega$, the sum over B' runs through subsets of $[\![1, \mu]\!]$ with $|B'| = \omega$. Notice that if a_k and a_m are in different connected components of $\mathcal{A}$, then $\text{dist}_{\mathcal{G}^{(\mathbb{T})}}(a_k, b_i) \leqslant R/2$ and $\text{dist}_{\mathcal{G}^{(\mathbb{T})}}(a_m, b_j) \leqslant R/2$ imply b_i and b_j are in different connected components of $\mathcal{A}$ (those of a_k and a_m, respectively), and in particular then $b_i \neq b_j$. As a consequence, the indices $j_1, \ldots j_\omega$ must be distinct, and in particular the random variables $b_{j_1}, \ldots, b_{j_\omega}$ are independent. Thus the previous

expression is bounded by

$$\sum_{A',B'} \mathbb{P}_{\mathcal{G}}\left(\mathrm{dist}_{\mathcal{G}^{(\mathbb{T})}}(a_{i_1},b_{j_1}) \leqslant R/2\right) \cdots \mathbb{P}_{\mathcal{G}}(\mathrm{dist}_{\mathcal{G}^{(\mathbb{T})}}(a_{i_\omega},b_{j_\omega}) \leqslant R/2)$$

$$\leqslant \binom{\mu}{\omega}^2 \left(\frac{4(d-1)^{R/2}}{N}\right)^\omega \ll N^{-\omega+\delta},$$

where we used that there are $\binom{\mu}{\omega}$ choices for A' and B' respectively, and the estimate (3.55) with $x = a_{i_1}, \ldots, a_{i_\omega}$. $\square$

Proof of (3.51). Similarly, to prove (3.51), by the union bound we have

$$\mathbb{P}_{\mathcal{G}}\left(|\{i \in [\![1,\mu]\!] : \mathrm{dist}_{\mathcal{G}^{(\mathbb{T})}}(b_i, \{a_j, b_k : j \in [\![1,\mu]\!], k \in [\![1,\mu]\!] \setminus \{i\}\}) \leqslant R/2\}| \geqslant 2\omega\right)$$

$$\leqslant \sum_{B'} \mathbb{P}_{\mathcal{G}}\left(\forall i \in B', \mathrm{dist}_{\mathcal{G}^{(\mathbb{T})}}(b_i, \{a_j, b_k : j \in [\![1,\mu]\!], k \in [\![1,\mu]\!] \setminus \{i\}\}) \leqslant R/2\right),$$

where B' runs through all subsets of $[\![1,\mu]\!]$ with $|B'| = 2\omega$. Next, we notice that, if for all $i \in B'$, we have $\mathrm{dist}_{\mathcal{G}^{(\mathbb{T})}}(b_i, \{a_j, b_k : j \in [\![1,\mu]\!], k \in [\![1,\mu]\!] \setminus \{i\}\}) \leqslant R/2$, then there must be subset $B'' \subset B'$ with $|B''| = \omega$ such that for all $i \in B''$, we have $\mathrm{dist}_{\mathcal{G}^{(\mathbb{T})}}(b_i, \{a_j, b_k : j \in [\![1,\mu]\!], k \in [\![1,\mu]\!] \setminus B''\}) \leqslant R/2$. By relabeling, without loss of generality, we assume that $B'' = \{\mu - \omega + 1, \mu - \omega + 2, \ldots, \mu\}$. Conditioned on $\vec{S}_1, \vec{S}_2, \ldots, \vec{S}_{\mu-\omega}$, we have

$$\mathbb{P}_{\mathcal{G}}\left(\mathrm{dist}(b_i, \{a_1, a_2, \ldots, a_\mu, b_1, b_2, \ldots, b_{\mu-\omega}\}) \leqslant R/2 \mid \vec{S}_1, \vec{S}_2, \ldots, \vec{S}_{\mu-\omega}\right) \leqslant 8\mu(d-1)^{R/2}/N,$$

for any $i \in [\![\mu - \omega + 1, \mu]\!]$. Therefore,

$$\sum_{B'} \mathbb{P}_{\mathcal{G}}\left(\forall i \in B', \mathrm{dist}_{\mathcal{G}^{(\mathbb{T})}}(b_i, \{a_j, b_k : j \in [\![1,\mu]\!], k \in [\![1,\mu]\!] \setminus \{i\}\}) \leqslant R/2\right)$$

$$\leqslant \sum_{B''} \mathbb{P}_{\mathcal{G}}\left(\forall i \in B'', \mathrm{dist}_{\mathcal{G}^{(\mathbb{T})}}(b_i, \{a_j, b_k : j \in [\![1,\mu]\!], k \in [\![1,\mu]\!] \setminus B''\}) \leqslant R/2\right)$$

$$\leqslant \binom{\mu}{\omega} \left(\frac{8\mu(d-1)^{R/2}}{N}\right)^\omega \ll N^{-\omega+\delta}.$$

since there are $\binom{\mu}{\omega}$ choices for B''. This completes the proof. $\square$

Proof of (3.52). By the assumption $\mathcal{G} \in \bar{\Omega}$, all except at most N^δ many vertices have radius-R tree neighborhoods. In particular, the same holds for $\mathcal{G}^{(\mathbb{T})}$. By (3.36), it follows that

$$\mathbb{P}_\mathcal{G} \text{ (the radius-}R \text{ neighborhood of } c_i \text{ contains cycles)} \leqslant 2N^{-1+\delta}.$$

By the union bound, and using that the number of ways to choose $\omega + 1$ elements from μ elements is bounded from above by $\mu^{\omega+1}$,

$$\mathbb{P}_\mathcal{G} \left(|\{i \in [\![1, \mu]\!] : \text{ radius-}R \text{ neighborhood of } c_i \text{ contains cycles}\}| \geqslant \omega + 1 \right)$$

$$\leqslant \mu^{\omega+1}(2N^{-1+\delta})^{\omega+1} \leqslant (2\mu)^{\omega+1} N^{-\omega-1+(\omega+1)\delta} \ll N^{-\omega+\delta},$$

given that $\delta < 1/\omega$ and using that $\mu \leqslant 2(d-1)^{\ell+1} = (\log N)^{O(1)}$ by the choice of parameters in Section 3.1. $\qquad\qquad\square$

3.3.9. *Proof of Proposition 3.16.*

Proof of (3.53). By the definition of the sets B_a and B_b, for any $i \in [\![1, \mu]\!] \setminus (\mathsf{B}_a \cup \mathsf{B}_b)$, we have

$$(3.56) \qquad\qquad \mathrm{dist}_{\mathcal{G}^{(\mathbb{T})}}(\{a_i, b_i\}, \{a_j, b_j : j \in [\![1, \mu]\!] \setminus \{i\}\}) > R/2.$$

Since $\tilde{\mathcal{G}}^{(\mathbb{T})}$ is obtained from $\mathcal{G}^{(\mathbb{T})}$ by removing the edges $\{b_i, c_i\}_{i \leqslant \nu}$ and adding the edges $\{a_i, b_i\}_{i \leqslant \nu}$, the claim (3.53) directly follows from (3.56). $\qquad\qquad\square$

Proof of (3.54). We consider three cases. If $\mathrm{dist}_{\tilde{\mathcal{G}}^{(\mathbb{T})}}(x, \{a_i, b_i\}) > R/4$ for all $i \in [\![1, \mu]\!]$, then the claim is trivial. If $\mathrm{dist}_{\tilde{\mathcal{G}}^{(\mathbb{T})}}(x, \{a_i, b_i\}) \leqslant R/4$ for some $i \in [\![1, \mu]\!] \setminus (\mathsf{B}_a \cup \mathsf{B}_b)$, then (3.53) implies that

$$\mathrm{dist}_{\tilde{\mathcal{G}}^{(\mathbb{T})}}(x, \{a_j, b_j\}) \geqslant \mathrm{dist}_{\tilde{\mathcal{G}}^{(\mathbb{T})}}(\{a_i, a_j\}, \{a_j, b_j\}) - \mathrm{dist}_{\tilde{\mathcal{G}}^{(\mathbb{T})}}(x, \{a_i, b_i\}) > R/2 - R/4 = R/4,$$

for any $j \in [\![1, \mu]\!] \setminus \{i\}$. Thus $|\{i \in [\![1, \mu]\!] : \mathrm{dist}_{\tilde{\mathcal{G}}^{(\mathbb{T})}}(x, \{a_i, b_i\}) \leqslant R/4\}| = 1 \leqslant 5\omega$ as claimed. In the remaining case, $\mathrm{dist}_{\tilde{\mathcal{G}}^{(\mathbb{T})}}(x, \{a_i, b_i\}) \leqslant R/4$ is only possible for $i \in \mathsf{B}_a \cup \mathsf{B}_b$. Therefore $|\{i \in [\![1, \mu]\!] : \mathrm{dist}_{\tilde{\mathcal{G}}^{(\mathbb{T})}}(x, \{a_i, b_i\}) \leqslant R/4\}| \leqslant |\mathsf{B}_a \cup \mathsf{B}_b| \leqslant 5\omega$ as claimed. $\qquad \square$

3.4. **The Green's function distance and switching cells.** The bounds provided in the Section 3.3.7 provide accurate control for distances at most $R/2$. However, *random* vertices are typically much further from each other, we require stronger upper bounds on the Green's function for such large distances. These bounds are in fact a general consequence of the *Ward identity*,

$$(3.57) \qquad \sum_j |G_{ij}(z)|^2 = \frac{\mathrm{Im}[G_{ii}(z)]}{\mathrm{Im}[z]},$$

which holds for the Green's function of any symmetric matrix (see (B.6)). To make use of it, we introduce a much coarser measure of distance in terms of the size of the Green's function as follows.

3.4.1. *Definition.* Given a parameter $M > 0$ (ultimately chosen in (3.60) below), we define a relation $\sim$ on $[\![N]\!] \setminus \mathbb{T}$ by setting $x \sim y$ if and only if

$$(3.58) \qquad \max_{\substack{u:\mathrm{dist}(x,u)\leqslant 4r, \\ v:\mathrm{dist}(y,v)\leqslant 4r}} |G_{uv}^{(\mathbb{T})}(z)| \geqslant \frac{M}{\sqrt{N\eta}},$$

where the distance in the maximum is with respect to the graph $\mathcal{G}^{(\mathbb{T})}$, and $\eta = \mathrm{Im}[z]$. The relation $\sim$ induces a graph $\mathcal{R}$ on the vertices $\{a_1, \ldots, a_\mu, b_1, \ldots, b_\nu\}$. We partition $\{a_1, \ldots, a_\mu, b_1, \ldots, b_\nu\}$ into its $\sim$-clusters. More precisely, we define $\mathbb{I}_1$ to be the vertex set consisting of the union of the connected components of $\mathcal{R}$ containing any element of $\{a_1, a_2, \ldots, a_\mu\}$, and we define $\mathbb{I}_2, \ldots, \mathbb{I}_\kappa$ be the vertex sets of the remaining connected components of $\mathcal{R}$.

Definition 3.17 (Cells)**.**
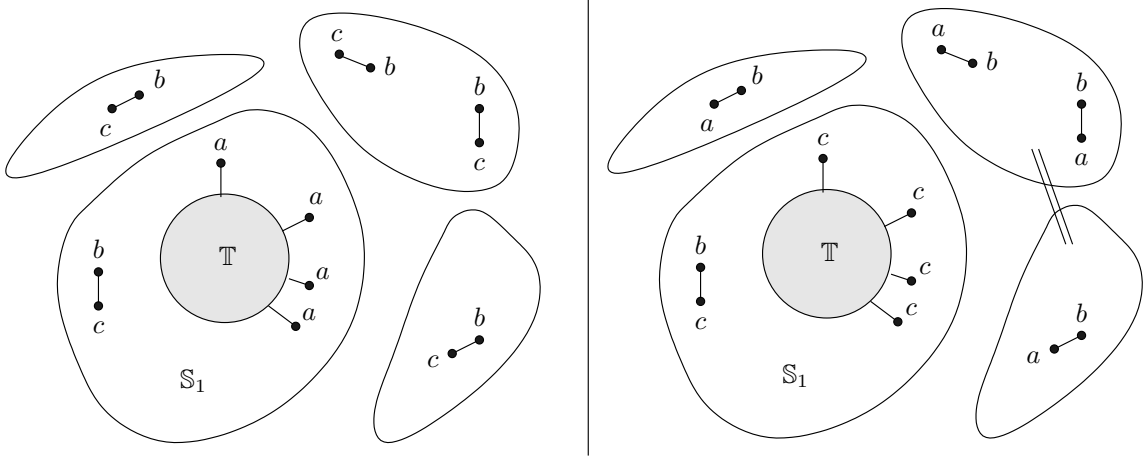
FIGURE 9. The S-cells are clusters of vertices that are close to one of the edges $\{b_i, c_i\}$ in the sense of the Green's function distance $\sim$. The S-cell $\mathbb{S}_1$ contains all a_i (the vertex boundary of $\mathcal{T}$ in the original graph) as well as those b_i which are close to any of the a_i in the sense of the Green's function distance. Since the switching may decrease distances between vertices, the S′-cells are defined by joining the S-cells which have vertices that are close to each other.

- *Define sets $\mathbb{S}_1, \mathbb{S}_2, \ldots, \mathbb{S}_\kappa \subset [\![1, N]\!]$ called* S-cells *by*

$$(3.59) \qquad \mathbb{S}_i = \mathbb{B}_{2r}(\mathbb{I}_i, \mathcal{G}^{(\mathbb{T})}).$$

  For any vertex $x \in [\![N]\!] \setminus \mathbb{T}$, we write $x \sim \mathbb{S}_i$ if there is $y \in \mathbb{I}_i$ such that $x \sim y$.

- *Define $\mathbb{S}'_1, \ldots, \mathbb{S}'_{\kappa'} \subset [\![1, N]\!]$ called* S′-cells *by combining the* S-*cells which are close to each other after switching: we set $\mathbb{S}'_1 = \mathbb{S}_1$ and join* S-*cells $\mathbb{S}_i$ and $\mathbb{S}_j$ with $i, j > 1$ if $\mathrm{dist}_{\tilde{\mathcal{G}}^{(\mathbb{T})}}(\mathbb{S}_i, \mathbb{S}_j) \leqslant 2r$.*

The S- and S′-cells are illustrated in Figure 9. The S-cells are defined in terms of the unswitched graph. In the switching process, distances between S-cells may decrease. This is accounted for by the coarser S′-cells. For later use, we note the following elementary properties of S-cells:

- For any $x \in \mathbb{S}_i$ and $y \in \mathbb{S}_j$ such that $i \neq j$ we have $|G_{xy}^{(\mathbb{T})}| < M/\sqrt{N\eta}$.
- For any vertex $x \in [\![N]\!] \setminus \mathbb{T}$, if $x \nsim \mathbb{S}_i$, then for any $y \in \mathbb{S}_i$, $|G_{xy}^{(\mathbb{T})}| < M/\sqrt{N\eta}$.
- If $b_k \in \mathbb{S}_i$, then also $c_k \in \mathbb{S}_i$; and, consequently, if $b_k \in \mathbb{S}'_i$ then $c_k \in \mathbb{S}'_i$.

3.4.2. *Estimates.* From now on, we fix the parameters M and ω' by

(3.60)
$$M = d^{9\ell}(\log N)^{\delta}, \quad \omega' = \lfloor \log N \rfloor,$$

where $\delta > 0$ was fixed at the beginning of Section 3. The next proposition shows that the cells do not cluster.

Proposition 3.18. *For any graph $\mathcal{G} \in \Omega_1^+(z, \ell)$ (as in Section 3.1.2), with probabililty at least $1 - \mathrm{o}(N^{-\omega+\delta})$ under $\mathbf{S}$, the following estimates hold:*

- *Any $x \in [\![N]\!] \setminus \mathbb{T}$ is $\sim$-connected to fewer than ω' of $\{b_1, b_2, \ldots, b_\mu\}$,*

(3.61)
$$|\{i \in [\![1, \mu]\!] : x \sim b_i\}| < \omega'.$$

  In particular, x is $\sim$-connected to at most ω' of the S-cells.

- *Except for at most ω' many indices i, the vertex b_i is a singleton in the graph $\mathcal{R}$, and thus the S-cell containing b_i is disjoint from $\{a_j, b_k : j \in [\![1, \mu]\!], k \in [\![1, \mu]\!] \setminus \{i\}\}$:*

(3.62)
$$|\{i \in [\![1, \mu]\!] : b_i \sim \{a_j, b_k : j \in [\![1, \mu]\!], k \in [\![1, \mu]\!] \setminus \{i\}\}\}| < \omega'.$$

  In particular, each S-cell contains at most ω' of $\{b_1, b_2, \ldots, b_\mu\}$.

- *Most S$'$-cells are far from the other vertices participating in the switching:*

(3.63)
$$\begin{aligned}
&|\{i \in [\![1, \nu]\!] : b_i \in \mathbb{S}'_j, \text{ such that } j = 1 \text{ or} \\
&\mathrm{dist}(\mathbb{S}'_j, \{a_k, b_m, c_m : k \in [\![1, \mu]\!] \setminus \{i\}, m \in [\![1, \nu]\!] \setminus \{i\}\}) \leqslant R/4\}| < \omega' + 5\omega.
\end{aligned}$$

  In particular, each S$'$-cell contains at most $\omega' + 5\omega$ of $\{b_1, b_2, \ldots, b_\nu\}$.

In the remainder of this section, we prove the above proposition. It is essentially a straightforward consequence of the definitions, combined with union bounds.

3.4.3. *Proof of Proposition 3.18.* The following two lemmas collect some elementary properties of the Green's function graph $\mathcal{R}$ on $\{a_1, \ldots, a_\mu, b_1, \ldots, b_\mu\}$ that we require.

Lemma 3.19. *Let $\mathcal{G} \in \Omega_1^+(z, \ell)$ (as in Section 3.1.2) and $x, y \in [\![N]\!] \setminus \mathbb{T}$. Then we have $x \not\sim y$ implies $\mathrm{dist}_{\mathcal{G}^{(\mathbb{T})}}(x, y) > 8r$.*

Proof. We show that $\mathrm{dist}_{\mathcal{G}^{(\mathbb{T})}}(x, y) \leqslant 8r$ implies $x \sim y$. Assume that $\mathrm{dist}_{\mathcal{G}^{(\mathbb{T})}}(x, y) \leqslant 8r$. Then there must be a vertex u such that $\mathrm{dist}_{\mathcal{G}^{(\mathbb{T})}}(x, u) \leqslant 4r$ and $\mathrm{dist}_{\mathcal{G}^{(\mathbb{T})}}(y, u) \leqslant 4r$. Moreover, by the definition (3.11) of $\Omega_1^+(z, \ell)$ and estimate (2.13), also $|G_{uu}^{(\mathbb{T})}(z)| \geqslant |m_{sc}(z)|/2 \geqslant M/\sqrt{N\eta}$, and thus $x \sim y$. $\quad\square$

Lemma 3.20. *Let $\mathcal{G} \in \Omega_1^+(z, \ell)$ (as in Section 3.1.2) and $x \in [\![N]\!] \setminus \mathbb{T}$. Then*

$$(3.64) \qquad \qquad \mathbb{P}_{\mathcal{G}}(b_i \sim x) \leqslant 16(d-1)^{8r}/M^2.$$

Proof. $\mathcal{G} \in \Omega_1^+(z, \ell)$ and (2.13) imply that $\mathrm{Im}[G_{xx}^{(\mathbb{T})}] \leqslant |G_{xx}^{(\mathbb{T})}| \leqslant 2$. Thus the Ward identity (B.6) implies

$$(3.65) \qquad \qquad \sum_i |G_{xi}^{(\mathbb{T})}|^2 = \mathrm{Im}[G_{xx}^{(\mathbb{T})}]/\eta \leqslant 2/\eta.$$

For any vertex $x \in [\![N]\!] \setminus \mathbb{T}$, set

$$\tilde{\mathbb{V}}_x := \left\{ i \in [\![N]\!] \setminus \mathbb{T} : |G_{xi}^{(\mathbb{T})}| \geqslant M/\sqrt{N\eta} \right\},$$

$$\mathbb{V}_x := \left\{ i \in [\![N]\!] \setminus \mathbb{T} : \mathrm{dist}_{\mathcal{G}^{(\mathbb{T})}}\left(i, \bigcup_{j \in \mathbb{B}_{4r}(x, \mathcal{G}^{(\mathbb{T})})} \tilde{\mathbb{V}}_j \right) \leqslant 4r \right\}.$$

The inequality (3.65) implies $|\tilde{\mathbb{V}}_x| \leqslant 2N/M^2$, and since any vertex has at most $2(d-1)^{4r}$ vertices in its radius-$4r$ neighborhood, we also have $|\mathbb{V}_x| \leqslant 8(d-1)^{8r}N/M^2$. Moreover, $i \notin \mathbb{V}_x$ implies that $i \not\sim x$. Thus

$$\mathbb{P}_{\mathcal{G}}(b_i \sim x) \leqslant \mathbb{P}_{\mathcal{G}}(b_i \in \mathbb{V}_x) \leqslant \frac{2}{N}|\mathbb{V}_x| \leqslant 16(d-1)^{8r}/M^2,$$

where the second inequality holds because b_i is approximately uniform (3.36). $\quad\square$

Proof of (3.61). The proof is similar to that of (3.49). By the union bound and (3.64), we have

$$\mathbb{P}(|\{i \in [\![1, \mu]\!] : b_i \sim x\}| \geqslant \omega') \leqslant \binom{\mu}{\omega'} \left(\frac{16(d-1)^{8r}}{M^2}\right)^{\omega'} \leqslant (\log N)^{-\delta \log N} \ll N^{-\omega},$$

where, in the second inequality, we used $\binom{\mu}{\omega'} \leqslant \mu^{\omega'}$ and that

$$16\mu d^{8r}/M^2 \leqslant 16 d^{\ell+1} d^{17\ell} d^{-18\ell} (\log N)^{-2\delta}$$

since $\mu \leqslant d^{\ell+1}$ and by the definition of M (3.60). $\qquad\square$

Proof of (3.62). The proof is similar to that of (3.51). Indeed, by the union bound and (3.64),

$$\mathbb{P}_{\mathcal{G}}\left(|\{i \in [\![1, \mu]\!] : b_i \sim \{a_j, b_k : j \in [\![1, \mu]\!], k \in [\![1, \mu]\!] \setminus \{i\}\}\}| \geqslant \omega'\right)$$

$$\leqslant \binom{\mu}{\omega'/2} \left(\frac{16(d-1)^{8r}\mu}{M^2}\right)^{\omega'/2} \leqslant (\log N)^{-\delta \log N/2} \ll N^{-\omega},$$

as needed. $\qquad\square$

Proof of (3.63). Recall the index sets $\mathsf{B}_a, \mathsf{B}_b \subset [\![1, \mu]\!]$ from (3.50), (3.51), and let $i \notin \mathsf{B}_a \cup \mathsf{B}_b$ be such that $b_i \nsim \{a_k, b_m : k \in [\![1, \mu]\!], m \in [\![1, \nu]\!] \setminus \{i\}\}$. Denote the S-cell containing b_i by $\mathbb{S}$; then $\mathbb{S}$ is not $\mathbb{S}_1$ and it is disjoint from $\{b_k : k \in [\![1, \mu]\!] \setminus \{i\}\}$.

By the definition of B_a, B_b and since b_j and c_j are adjacent in $\mathcal{G}^{(\mathbb{T})}$ we have

$$(3.66) \quad \mathrm{dist}_{\mathcal{G}^{(\mathbb{T})}}(\{a_i, b_i, c_i\}, \{a_k, b_m, c_m : k \in [\![1, \mu]\!] \setminus \{i\}, m \in [\![1, \nu]\!] \setminus \{i\}\}) \geqslant R/2 - 2.$$

Since the graph $\tilde{\mathcal{G}}^{(\mathbb{T})}$ is obtained from $\mathcal{G}^{(\mathbb{T})}$ by removing edges $\{b_j, c_j\}_{j \leqslant \nu}$ and adding edges $\{a_j, b_j\}_{j \leqslant \nu}$, we also have

$$(3.67) \quad \mathrm{dist}_{\tilde{\mathcal{G}}^{(\mathbb{T})}}(\{a_i, b_i, c_i\}, \{a_k, b_m, c_m : k \in [\![1, \mu]\!] \setminus \{i\}, m \in [\![1, \nu]\!] \setminus \{i\}\}) \geqslant R/2 - 2.$$

Moreover, for any other S-cell $\mathbb{S}_j \neq \mathbb{S}, \mathbb{S}_1$, we have

$$\mathrm{dist}_{\tilde{\mathcal{G}}^{(\mathbb{T})}}(\mathbb{S}, \mathbb{S}_j) \geqslant -2r + \mathrm{dist}_{\tilde{\mathcal{G}}^{(\mathbb{T})}}(b_i, \mathbb{I}_j) - 2r \geqslant R/2 - 2 - 4r > 2r,$$

where we used (3.67), $r \ll R$ and the definition (3.59) of S-cells, i.e. $\mathbb{S} = \mathbb{B}_{2r}(b_i, \mathcal{G}^{(\mathbb{T})})$ and $\mathbb{S}_j = \mathbb{B}_{2r}(\mathbb{I}_j, \mathcal{G}^{(\mathbb{T})})$. Thus $\mathbb{S}$ is a S$'$-cell itself, and

$$\mathrm{dist}_{\tilde{\mathcal{G}}^{(\mathbb{T})}}(\mathbb{S}, \{a_k, b_m, c_m : k \in [\![1, \mu]\!] \setminus \{i\}, m \in [\![1, \nu]\!] \setminus \{i\}\})$$

$$\geqslant \mathrm{dist}_{\tilde{\mathcal{G}}^{(\mathbb{T})}}(b_i, \{a_k, b_m, c_m : k \in [\![1, \mu]\!] \setminus \{i\}, m \in [\![1, \nu]\!] \setminus \{i\}\}) - 2r \geqslant R/2 - 2r - 2 > R/4,$$

where we used $r \ll R$. Therefore, only $i \in \mathsf{B}_a \cup \mathsf{B}_b$ or $b_i \sim \{a_k, b_m : k \in [\![1, \mu]\!], m \in [\![1, \nu]\!] \setminus \{i\}\}$ contribute to the statement (3.63). Thus, combining (3.62) with the estimate $|\mathsf{B}_a \cup \mathsf{B}_b| \leqslant 5\omega$ from (3.50), (3.51), and with (3.62), the estimate (3.63) follows. $\qquad\square$

3.5. **Stability under removal of a neighborhood.** The following deterministic estimate shows that removing the neighborhood $\mathbb{T}$ from the graph $\mathcal{G}$ has a small effect on the Green's function in the complement of $\mathbb{T}$.

Proposition 3.21. *Let $z \in \mathbb{C}_+$ and $\sqrt{d-1} \geqslant (\omega + 1)^2 2^{2\omega + 10}$, and let $\mathcal{G} \in \bar{\Omega}$ (as in Section 3.1.2) be a graph such that, for all $i, j \in [\![N]\!]$,*

$$(3.68) \qquad\qquad |G_{ij} - P_{ij}(\mathcal{E}_r(i, j, \mathcal{G}))| \leqslant |m_{sc}|q^r.$$

Then, for all vertices $i, j \in [\![N]\!] \setminus \mathbb{T}$, we have

$$(3.69) \qquad\qquad |G_{ij}^{(\mathbb{T})} - P_{ij}(\mathcal{E}_r(i, j, \mathcal{G}^{(\mathbb{T})}))| \leqslant 2|m_{sc}|q^r.$$

As discussed in Section 3.1, the removal of $\mathbb{T}$ is useful because our switchings have a smaller effect in $\mathcal{G}^{(\mathbb{T})}$ than they do in $\mathcal{G}$. Indeed, in the original graph $\mathcal{G}$, our switchings have the effect of removing two edges and adding two edges, while in $\mathcal{G}^{(\mathbb{T})}$ our switchings only remove the edges $\{b_i, c_i\}_{i \leqslant \nu}$ and add the edges $\{a_i, b_i\}_{i \leqslant \nu}$. In the

next few sections, we therefore work with $\mathcal{G}^{(\mathbb{T})}$ and its switched version $\tilde{\mathcal{G}}^{(\mathbb{T})}$, and only return to the full graph in Section 3.8.

The remainder of this section is devoted to the proof of the proposition. The main ingredients are that (i) given any i, j, there can only be a few vertices in $\mathbb{T}$ that are close to i or j, by the deterministic assumption on the excess of R-neigborhoods, and (ii) that for all other vertices in $\mathbb{T}$, the decay of the Green's function implied by (3.68) shows that the removal of them has a small effect.

3.5.1. *Step 1: Removal of vertices close to i or j.* From (3.47), recall that $\mathbb{T}_\ell = \{v \in \mathcal{G} : \text{dist}_\mathcal{G}(1, v) = \ell\}$ is the set of inner vertex boundary of $\mathcal{T}$. The first step of the proof of Proposition 3.21 consists of removing the vertices in $\mathbb{T}_\ell$ that are close to i or j. The set of such vertices is

$$(3.70) \qquad \mathbb{U} = \{v \in \mathbb{T}_\ell : \text{dist}_{\mathcal{G}\setminus\mathcal{T}}(i, v) \leqslant r\} \cup \{v \in \mathbb{T}_\ell : \text{dist}_{\mathcal{G}\setminus\mathcal{T}}(j, v)\} \leqslant r\},$$

where $\mathcal{G} \setminus \mathcal{T}$ is obtained from $\mathcal{G}$ by removing the subgraph $\mathcal{T}$ induced by $\mathcal{G}$ on $\mathbb{T}$ (but not removing $\mathbb{T}_\ell$). Then $|\mathbb{U}| \leqslant 2\omega + 2$ by (3.31). The following proposition shows that the Green's function remains to be locally approximated after removing $\mathbb{U}$.

Proposition 3.22. *Under the assumptions of Proposition 3.21, for any vertex set $\mathbb{U} \subset \mathbb{T}$ with $|\mathbb{U}| \leqslant 2\omega + 2$,*

$$(3.71) \qquad\qquad |G_{ij}^{(\mathbb{U})} - P_{ij}(\mathcal{E}_r(i, j, \mathcal{G}^{(\mathbb{U})}))| \leqslant 3|m_{sc}|q^r/2.$$

The proof of Proposition 3.22 follows a general structure that occurs repeatedly in similar estimates throughout the paper.

(i) The first ingredient in this structure, which we refer to as *localization*, replaces the Green's function $P_{ij}(\mathcal{E}_r(i, j, \mathcal{G}))$ of the vertex-dependent graph $\mathcal{E}_r(i, j, \mathcal{G})$ by the Green's function $P_{ij} = P_{ij}(\mathcal{G}_0)$ of a graph $\mathcal{G}_0$ that does not depend on
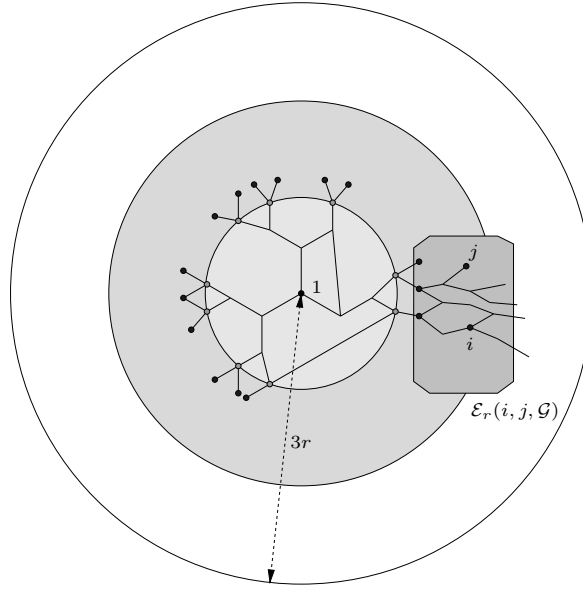
FIGURE 10. The innermost disk shows $\mathbb{T}$, the second largest disk the set $\mathbb{X}$, and the outermost disk $\mathcal{G}_0$. For any $i, j \in \mathbb{X}$, the graph $\mathcal{E}_r(i, j, \mathcal{G})$ is contained in $\mathcal{G}_0$.

i, j, by an application of Remark 2.8. For this, among other things, we need to verify the assumptions of Proposition 3.22.

(ii) The second ingredient, which we refer to as the *starting point* for the argument, is an algebraic relation that expresses the quantity to be estimated in a convenient form. The starting point typically follows from the Schur complement formula or the resolvent formula.

(iii) The third ingredient is a collection of previously established estimates required to estimate the expressions given by the starting point. It typically includes estimates on elements of Green's functions and graph distances.

The actual proofs then usually follow by combination of the above ingredients. In principle, this step is straightforward, but often several different cases need to be distinguished, which makes some of the arguments appear somewhat lengthy.

Below we provide the first instance of the strategy described above to prove Proposition 3.22.

Localization. We approximate $P_{ij}(\mathcal{E}_r(i,j,\mathcal{G}))$ by a vertex independent Green's function P_{ij} according to Remark 2.8, applied with $\mathcal{G}_0 = \mathcal{B}_{3r}(1,\mathcal{G})$ and $\mathbb{X} = \mathbb{B}_{2r}(\mathbb{U},\mathcal{G})$. We abbreviate

$$\mathcal{G}_1 = \mathrm{TE}(\mathcal{G}_0), \quad P = G(\mathcal{G}_1), \qquad \mathcal{G}_1^{(\mathbb{U})} = \mathrm{TE}(\mathcal{G}_0^{(\mathbb{U})}), \quad P^{(\mathbb{U})} = G(\mathcal{G}_1^{(\mathbb{U})}).$$

Verification of assumptions in Proposition 2.7. As subgraphs of $\mathcal{G} \in \bar{\Omega}$, the radius-$R$ neighborhoods of $\mathcal{G}_0$ and $\mathcal{G}_0^{(\mathbb{U})}$ have excess at most ω. By convention, the deficit function of $\mathcal{G}_0$ vanishes, on each connected component of $\mathcal{G}_0^{(\mathbb{U})}$, the deficit function of $\mathcal{G}^{(\mathbb{U})}$ obeys $\sum g(v) \leqslant \omega + (2\omega + 2) \leqslant 8\omega$, by Proposition 3.11. Thus the assumptions for (2.14) are verified for both graphs, and for any $i, j \in \mathbb{X}$,

(3.72)
$$\left|P_{ij}(\mathcal{E}_r(i,j,\mathcal{G})) - P_{ij}\right| \leqslant 2^{2\omega+3}|m_{sc}|q^{r+1}, \qquad \left|P_{ij}(\mathcal{E}_r(i,j,\mathcal{G}^{(\mathbb{U})})) - P_{ij}^{(\mathbb{U})}\right| \leqslant 2^{2\omega+3}|m_{sc}|q^{r+1}$$

provided that $\sqrt{d-1} \geqslant 2^{\omega+2}$.

Starting point. To remove $\mathbb{U}$, we apply the Schur complement formula (B.4): for any $i, j \in \mathcal{G}^{(\mathbb{U})}$,

(3.73)
$$G_{ij} - G_{ij}^{(\mathbb{U})} = \sum_{x,y \in \mathbb{U}} G_{ix}(G|_{\mathbb{U}})_{xy}^{-1}G_{yj},$$

$$P_{ij} - P_{ij}^{(\mathbb{U})} = \sum_{x,y \in \mathbb{U}} P_{ix}(P|_{\mathbb{U}})_{xy}^{-1}P_{yj}.$$

Our goal is to show that the difference $G_{ij}^{(\mathbb{U})} - P_{ij}^{(\mathbb{U})}$ is small, by using that the difference of G and P is small. As evident from the right-hand sides of (3.73), for this we require upper bounds on the entries of G and $(G|_{\mathbb{U}})^{-1}$ (and analogously for P and $(P|_{\mathbb{U}})^{-1}$).

Green's function estimates. By assumption (3.68) and (2.12)–(2.13), we have

$$
(3.74) \qquad
\begin{cases}
|G_{xx}| \geqslant |m_d| - |m_{sc}|/4 - |m_{sc}|q^r \geqslant 3|m_{sc}|/5, & \\
|G_{xw}| \leqslant 2^{\omega+2}|m_{sc}|q + |m_{sc}|q^r & (x \neq w), \\
|G_{xw}| \leqslant |m_{sc}|q^r & (\mathrm{dist}_{\mathcal{G}}(x, w) > r).
\end{cases}
$$

These bounds imply the upper bounds for the entries of $(G|_{\mathbb{U}})^{-1}$ stated in the following claim. The claim essentially follows from the fact that the off-diagonal entries of $G|_{\mathbb{U}}$ are much smaller than the diagonal entries which have size roughly m_{sc}.

Claim 3.23. *Under the assumptions of Proposition 3.21, for any $\mathbb{U} \subset \mathbb{T}$ with $|\mathbb{U}| \leqslant 2\omega + 2$, and any $x, y \in \mathbb{U}$,*

$$
(3.75) \qquad |(G|_{\mathbb{U}})^{-1}_{xy}| \leqslant 2/|m_{sc}|, \quad |(P|_{\mathbb{U}})^{-1}_{xy}| \leqslant 2/|m_{sc}|.
$$

Proof. By the identity $G|_{\mathbb{U}}(G|_{\mathbb{U}})^{-1} = I_{\mathbb{U} \times \mathbb{U}}$, we have

$$
(3.76) \qquad \delta_{xy} = G_{xx}(G|_{\mathbb{U}})^{-1}_{xy} + \sum_{w \in \mathbb{U} \setminus \{x\}} G_{xw}(G|_{\mathbb{U}})^{-1}_{wy}.
$$

Let $\Gamma := \max_{x,y \in \mathbb{U}} |(G|_{\mathbb{U}})^{-1}_{xy}|$. Then (3.74) and (3.76) imply

$$
|G_{xx}||(G|_{\mathbb{U}})^{-1}_{xy}| \leqslant \delta_{xy} + \sum_{w \in \mathbb{U} \setminus \{x\}} |G_{xw}|\Gamma \leqslant 1 + (2^{\omega+2}q + q^r)|\mathbb{U}||m_{sc}|\Gamma.
$$

Taking the maximum over $x, y \in \mathbb{U}$ in the equation above and using (3.74) gives

$$
\Gamma \leqslant \frac{5}{3|m_{sc}|} + \frac{5}{3}(2^{\omega+2}q + q^r)|\mathbb{U}|\Gamma \leqslant \frac{5}{3|m_{sc}|} + \frac{\Gamma}{6},
$$

provided that $\sqrt{d-1} \geqslant (\omega + 1)2^{\omega+6}$. $\Gamma \leqslant 2/|m_{sc}|$ follows by rearranging. The same argument applies to $P|_{\mathbb{U}}$, and we obtain (3.75). $\qquad \square$

Proof of Proposition 3.22. First consider the case that at least one of i and j is not in $\mathbb{X}$ (i.e. far from $\mathbb{U}$). Then $\mathcal{E}_r(i,j,\mathcal{G}) = \mathcal{E}_r(i,j,\mathcal{G}^{(\mathbb{U})})$, and (3.71) follows directly from

$$|G_{ij} - G_{ij}^{(\mathbb{U})}| = \left| \sum_{x,y \in \mathbb{U}} G_{ix}(G|_{\mathbb{U}})_{xy}^{-1} G_{yj} \right| \leqslant (2^{\omega+2}q + q^r)q^r |\mathbb{U}|^2 |m_{sc}|^2 (2/|m_{sc}|) \leqslant |m_{sc}|q^r/2,$$

where we used (3.74), (3.75) and that $\sqrt{d-1} \geqslant (\omega+1)^2 2^{\omega+7}$.

Next consider the main case $i,j \in \mathbb{X}$. By (3.72), it suffices to bound the right-hand side of

$$(3.77) \qquad |G_{ij}^{(\mathbb{U})} - P_{ij}^{(\mathbb{U})}| \leqslant |G_{ij} - P_{ij}| + \sum_{x,y \in \mathbb{U}} \left| G_{ix}(G|_{\mathbb{U}})_{xy}^{-1} G_{yj} - P_{ix}(P|_{\mathbb{U}})_{xy}^{-1} P_{yj} \right|,$$

which follows from taking difference of expressions in (3.73). By (3.68) and (3.72), since for all vertices $i,j \in \mathbb{X}$ and $x,y \in \mathbb{U} \subset \mathbb{T}$, we have $\mathcal{E}_r(i,j,\mathcal{G}), \mathcal{E}_r(i,x,\mathcal{G}), \mathcal{E}_r(y,j,\mathcal{G}), \mathcal{E}_r(x,y,\mathcal{G}) \subset \mathcal{G}_0$,

$$(3.78) \quad |G_{ij} - P_{ij}|, |G_{ix} - P_{ix}|, |G_{yj} - P_{yj}|, |G_{xy} - P_{xy}| \leqslant |m_{sc}|q^r + 2^{2\omega+3}|m_{sc}|q^{r+1}.$$

Together with (3.75) and the resolvent formula (B.1), it follows that

(3.79)
$$|(G|_{\mathbb{U}})_{xy}^{-1} - (P|_{\mathbb{U}})_{xy}^{-1}| = |[(G|_{\mathbb{U}})^{-1}(G|_{\mathbb{U}} - P|_{\mathbb{U}})(P|_{\mathbb{U}})^{-1}]_{xy}| \leqslant 4|\mathbb{U}|^2(1 + 2^{2\omega+3}q)q^r/|m_{sc}|.$$

Using (3.74), (3.75), (3.78), and (3.79), the sum on the right-hand side of (3.77) is bounded by

$$\sum_{x,y \in \mathbb{U}} \left(|G_{ix} - P_{ix}| \, |(G|_{\mathbb{U}})_{xy}^{-1} G_{yj}| + |P_{ix}(P|_{\mathbb{U}})_{xy}^{-1}||G_{yj} - P_{yj}| + |P_{ix}| \, |(G|_{\mathbb{U}})_{xy}^{-1} - (P|_{\mathbb{U}})_{xy}^{-1}||G_{yj}| \right)$$

(3.80)
$$\leqslant 4|m_{sc}|q^r(1 + 2^{2\omega+3}q) \left(|\mathbb{U}|^2(2^{\omega+2}q + q^r) + |\mathbb{U}|^4(2^{\omega+2}q + q^r)^2 \right) \leqslant |m_{sc}|q^r/4$$

where we used that $\sqrt{d-1} \geqslant (\omega+1)^2 2^{2\omega+10}$. The claim follows by combining this bound for (3.77) with (3.72). $\qquad \square$

3.5.2. *Step 2: Estimate of $G_{ij}^{(\mathbb{T})}$ using $G_{ij}^{(\mathbb{U})}$.* Next we pass from $G_{ij}^{(\mathbb{U})}$ to $G_{ij}^{(\mathbb{T})}$. By definition of $\mathbb{U}$, there are no vertices in $\mathbb{T} \setminus \mathbb{U}$ that are close to i or j in the graph $\mathcal{G}^{(\mathbb{T})}$. Thus the step mostly follows from the decay of the Green's function together with the assumption that there are few cycles.

Starting point. Define $\mathcal{G}_0 = \mathcal{B}_{3r}(1, \mathcal{G})$ and $\mathcal{G}_1 = \mathrm{TE}(\mathcal{G}_0)$ as in Section 3.5.1. The normalized adjacency matrices of $\mathcal{G}^{(\mathbb{U})}$ and $\mathcal{G}_1^{(\mathbb{U})} = \mathrm{TE}(\mathcal{G}_0^{(\mathbb{U})})$ have the block matrix form

$$\begin{bmatrix} H^{(\mathbb{U})} & B' \\ B & D \end{bmatrix}, \quad \begin{bmatrix} H^{(\mathbb{U})} & B_1' \\ B_1 & D_1 \end{bmatrix},$$

where $H^{(\mathbb{U})}$ is the normalized adjacency matrix of $\mathcal{T}^{(\mathbb{U})}$. The nonzero entries of B and B_1 occur for the indices $(i, j) \in \{a_1, \ldots, a_\mu\} \times \mathbb{T}_\ell \setminus \mathbb{U}$ and take values $1/\sqrt{d-1}$. Notice that $\tilde{B}_{ij} = (\tilde{B}_1)_{ij}$. We denote the normalized Green's functions of $\mathcal{G}^{(\mathbb{U})}$ and $\mathcal{G}_1^{(\mathbb{U})}$ by $G^{(\mathbb{U})}$ and $P^{(\mathbb{U})}$ respectively. By the Schur complement formula (B.4), for any $i, j \in [\![N]\!] \setminus \mathbb{T}$,

$$(3.81) \qquad G_{ij}^{(\mathbb{T})} = (D - z)_{ij}^{-1} = G_{ij}^{(\mathbb{U})} - \sum_{x,y \in \mathbb{T} \setminus \mathbb{U}} G_{ix}^{(\mathbb{U})} (G^{(\mathbb{U})}|_{\mathbb{T} \setminus \mathbb{U}})_{xy}^{-1} G_{yj}^{(\mathbb{U})},$$

and also

$$(3.82) \qquad (P^{(\mathbb{U})}|_{\mathbb{T} \setminus \mathbb{U}})^{-1} = H^{(\mathbb{U})} - z - B_1'(D_1 - z)^{-1} B_1,$$

$$(3.83) \qquad (G^{(\mathbb{U})}|_{\mathbb{T} \setminus \mathbb{U}})^{-1} = H^{(\mathbb{U})} - z - B'(D - z)^{-1} B.$$

Claim 3.24. *For any $x \in \mathbb{T}_\ell \setminus \mathbb{U}$,*

$$(3.84) \qquad \sum_{y \in \mathbb{T} \setminus \mathbb{U}} |(P^{(\mathbb{U})}|_{\mathbb{T} \setminus \mathbb{U}})_{xy}^{-1}| \leqslant 2(|z| + 1).$$

Proof. For any $x \in \mathbb{T}_\ell \setminus \mathbb{U}$, by (3.82) we have

$$\sum_{y \in \mathbb{T} \setminus \mathbb{U}} |(P^{(\mathbb{U})}|_{\mathbb{T} \setminus \mathbb{U}})^{-1}_{xy}| = \sum_{y \in \mathbb{T} \setminus \mathbb{U}} |(H^{(\mathbb{U})} - z - B_1'(D_1 - z)^{-1} B_1)_{xy}|$$

$$\leqslant \sum_{y \in \mathbb{T} \setminus \mathbb{U}} H^{(\mathbb{U})}_{xy} + |z| + \sum_{y \in \mathbb{T} \setminus \mathbb{U}} |(B_1'(D_1 - z)^{-1} B_1)_{xy}|$$

$$(3.85) \qquad \leqslant \frac{\omega + 1}{\sqrt{d-1}} + |z| + \sum_{y \in \mathbb{T} \setminus \mathbb{U}} |(B_1'(D_1 - z)^{-1} B_1)_{xy}|.$$

In the last inequality, we used that the excess of $\mathcal{T}^{(\mathbb{U})}$ is at most ω so that for any $x \in \mathbb{T}_\ell \setminus \mathbb{U}$, we have $\deg_{\mathcal{T}^{(\mathbb{U})}}(x) \leqslant \omega + 1$ and thus $\sum_{y \in \mathbb{T} \setminus \mathbb{U}} H^{(\mathbb{U})}_{xy} \leqslant (\omega + 1)/\sqrt{d-1}$. The terms in the last sum in (3.85) vanish unless $y \in \mathbb{T}_\ell \setminus \mathbb{U}$. Therefore the sum is bounded by

$$\sum_{\substack{i \in [\![1,\mu]\!] \\ l_i = x}} (B_1')_{l_i a_i} |(D_1 - z)^{-1}_{a_i a_i}| (B_1)_{a_i l_i} + \sum_{\substack{i \neq j \in [\![1,\mu]\!] \\ l_i, l_j \in \mathbb{T}_\ell \setminus \mathbb{U}}} (B_1')_{l_i a_i} |(D_1 - z)^{-1}_{a_i a_j}| (B_1)_{a_j l_j}$$

$$(3.86) \qquad \leqslant \frac{1}{d-1} \sum_{i \in [\![1,\mu]\!]} \mathbb{1}_{x = l_i} |(D_1 - z)^{-1}_{a_i a_i}| + \frac{1}{d-1} \sum_{i \neq j \in [\![1,\mu]\!]} |(D_1 - z)^{-1}_{a_i a_j}|.$$

For the first sum in (3.86), the number of vertices a_i adjacent to x is at most $d - 1$. For the second sum, by (3.29), for all pairs $i \neq j$ with at most $(2\omega)^2$ exceptions, $\operatorname{dist}_{\mathcal{G}^{(\mathbb{T})}}(a_i, a_j) > R/2$. For these pairs, a_i and a_j are in different connected components of the graph $\mathcal{G}_1^{(\mathbb{T})}$ which means that $|(D_1 - z)^{-1}_{a_i a_j}| = 0$. Therefore there are at most $(2\omega)^2$ non-vanishing terms in the second sum. We use also that

$$|(D_1 - z)^{-1}_{a_i a_j}| = P_{a_i a_j}(\operatorname{TE}(\mathcal{G}_0^{(\mathbb{T})})) \leqslant 3|m_{sc}|/2,$$

which follows from (2.15), provided that $\sqrt{d-1} \geqslant 2^{2\omega+3}$. Therefore,

$$(3.87) \qquad (3.86) \leqslant \left(1 + \frac{(2\omega)^2}{d-1}\right) \max_{i,j \in [\![1,\mu]\!]} |(D_1 - z)^{-1}_{a_i a_j}| \leqslant \frac{3}{2} |m_{sc}| \left(1 + \frac{(2\omega)^2}{d-1}\right),$$

By combining (3.85) and (3.87), we have shown that

$$\sum_{y \in \mathbb{T} \setminus \mathbb{U}} |(P^{(\mathbb{U})}|_{\mathbb{T} \setminus \mathbb{U}})_{xy}^{-1}| \leqslant \frac{\omega + 1}{\sqrt{d-1}} + |z| + \frac{3}{2}|m_{sc}| \left(1 + \frac{(2\omega)^2}{d-1} \right) \leqslant 2(|z|+1),$$

provided that $\sqrt{d-1} \geqslant 8(\omega + 1)$. This completes the proof. $\qquad \square$

Claim 3.25. *For any $x, y \in \mathbb{T} \setminus \mathbb{U}$,*

(3.88) $$|G_{xy}^{(\mathbb{U})} - P_{xy}^{(\mathbb{U})}| \leqslant 2|m_{sc}|q^r,$$

(3.89) $$|G^{(\mathbb{U})}|_{\mathbb{T} \setminus \mathbb{U}})_{xy}^{-1} - (P^{(\mathbb{U})}|_{\mathbb{T} \setminus \mathbb{U}})_{xy}^{-1}| \leqslant 48(|z|+1)q^r,$$

Proof. Define matrices $\mathcal{W}$ and $\mathcal{E}$ by

(3.90) $$G^{(\mathbb{U})}|_{\mathbb{T} \setminus \mathbb{U}} = P^{(\mathbb{U})}|_{\mathbb{T} \setminus \mathbb{U}} + \mathcal{W},$$

(3.91) $$(G^{(\mathbb{U})}|_{\mathbb{T} \setminus \mathbb{U}})^{-1} = (P^{(\mathbb{U})}|_{\mathbb{T} \setminus \mathbb{U}})^{-1} + \mathcal{E}.$$

From (3.71), (3.72), for any $x, y \in \mathbb{T} \setminus \mathbb{U}$, we have $|\mathcal{W}_{xy}| \leqslant 2|m_{sc}|q^r$. We claim the same estimate holds for the entries of the matrix $\mathcal{E}$. Notice from (3.82), (3.83) that $\mathcal{E}_{xy} \neq 0$ only for $x, y \in \mathbb{T}_\ell \setminus \mathbb{U}$. Let $\Gamma := \max_{x,y \in \mathbb{T} \setminus \mathbb{U}} |\mathcal{E}_{xy}| = \max_{x,y \in \mathbb{T}_\ell \setminus \mathbb{U}} |\mathcal{E}_{xy}|$. By taking the product of (3.90) and (3.91),

(3.92) $$\mathcal{E} + (P^{(\mathbb{U})}|_{\mathbb{T} \setminus \mathbb{U}})^{-1}\mathcal{W}\mathcal{E} + (P^{(\mathbb{U})}|_{\mathbb{T} \setminus \mathbb{U}})^{-1}\mathcal{W}(P^{(\mathbb{U})}|_{\mathbb{T} \setminus \mathbb{U}})^{-1} = 0.$$

For any $x, y \in \mathbb{T} \setminus \mathbb{U}$, therefore

$$|\mathcal{E}_{xy}| \leqslant \sum_{i,j \in \mathbb{T} \setminus \mathbb{U}} |(P^{(\mathbb{U})}|_{\mathbb{T} \setminus \mathbb{U}})_{xi}^{-1}||\mathcal{W}_{ij}||\mathcal{E}_{jy}| + \sum_{i,j \in \mathbb{T} \setminus \mathbb{U}} |(P^{(\mathbb{U})}|_{\mathbb{T} \setminus \mathbb{U}})_{xi}^{-1}||\mathcal{W}_{ij}||(P^{(\mathbb{U})}|_{\mathbb{T} \setminus \mathbb{U}})_{jy}^{-1}|$$

$$\leqslant |\mathbb{T} \setminus \mathbb{U}|(2|m_{sc}|q^r)\Gamma \sum_{i \in \mathbb{T} \setminus \mathbb{U}} |(P^{(\mathbb{U})}|_{\mathbb{T} \setminus \mathbb{U}})_{xi}^{-1}| + (2|m_{sc}|q^r) \sum_{i \in \mathbb{T} \setminus \mathbb{U}} |(P^{(\mathbb{U})}|_{\mathbb{T} \setminus \mathbb{U}})_{xi}^{-1}| \sum_{j \in \mathbb{T} \setminus \mathbb{U}} |(P^{(\mathbb{U})}|_{\mathbb{T} \setminus \mathbb{U}})_{jy}^{-1}|$$

$$\leqslant 4(|z|+1)(d-1)^\ell (2|m_{sc}|q^r)\Gamma + 4(|z|+1)^2 (2|m_{sc}|q^r)$$

$$\leqslant \Gamma/2 + 4(|z|+1)^2 (2|m_{sc}|q^r).$$

For the second inequality, we used $|\mathcal{W}_{xy}| \leqslant 2|m_{sc}|q^r$; for the third inequality we used $|\mathbb{T} \setminus \mathbb{U}| \leqslant |\mathbb{T}| \leqslant 1 + d + d(d-1) + \cdots + d(d-1)^{\ell-1} \leqslant 2(d-1)^\ell$, and (3.84); for the last inequality, we used $r = 2\ell + 1$, so that $(d-1)^\ell q^r \leqslant (d-1)^{-1/2}$ and $|zm_{sc}| \leqslant 2$. Taking the maximum on the right-hand side of the above inequality, and rearranging, we get

$$\Gamma \leqslant 16(|z| + 1)^2 |m_{sc}|q^r \leqslant 48(|z| + 1)q^r,$$

as claimed. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proof of Proposition 3.21. To prove the proposition, we define $\mathbb{U}$ by (3.70), and show that

$$(3.93) \qquad\qquad\qquad |G_{ij}^{(\mathbb{T})} - G_{ij}^{(\mathbb{U})}| \leqslant |m_{sc}|q^r/4.$$

This implies the claim. Indeed, the definition of $\mathbb{U}$ implies that $\mathcal{E}_r(i, j, \mathcal{G}^{(\mathbb{U})}) = \mathcal{E}_r(i, j, \mathcal{G}^{(\mathbb{T})})$, and therefore (3.69) follows from (3.72), (3.93) and Proposition 3.22:

$$|G_{ij}^{(\mathbb{T})} - P_{ij}(\mathcal{E}_r(i, j, \mathcal{G}^{(\mathbb{T})}))| \leqslant |G_{ij}^{(\mathbb{T})} - G_{ij}^{(\mathbb{U})}| + |G_{ij}^{(\mathbb{U})} - P_{ij}^{(\mathbb{U})}| + |P_{ij}(\mathcal{E}_r(i, j, \mathcal{G}^{(\mathbb{U})})) - P_{ij}^{(\mathbb{U})}|$$

$$\leqslant |m_{sc}|q^r/4 + 3|m_{sc}|q^r/2 + 2^{2\omega+3}|m_{sc}|q^{r+1} \leqslant 2|m_{sc}|q^r.$$

Thus it remains to prove (3.93). By definition of $\mathbb{U}$, we have $\text{dist}_{\mathcal{G}^{(\mathbb{U})}}(\{i, j\}, \mathbb{T} \setminus \mathbb{U}) > r$, and therefore Proposition 3.22 implies

$$(3.94) \qquad\qquad \max_{x \in \mathbb{T} \setminus \mathbb{U}} \left\{ |G_{ix}^{(\mathbb{U})}|, |G_{jx}^{(\mathbb{U})}| \right\} \leqslant 3|m_{sc}|q^r/2 \leqslant 2|m_{sc}|q^r.$$

Furthermore, by (3.81),

$$(3.95) \qquad \begin{aligned} |G_{ij}^{(\mathbb{T})} - G_{ij}^{(\mathbb{U})}| &\leqslant \sum_{x,y \in \mathbb{T} \setminus \mathbb{U}} |G_{ix}^{(\mathbb{U})}(H^{(\mathbb{U})} - z - B'G^{(\mathbb{T})}B)_{xy}G_{yj}^{(\mathbb{U})}| \\ &\leqslant 4|m_{sc}|^2 q^{2r} \sum_{x,y \in \mathbb{T} \setminus \mathbb{U}} |(H^{(\mathbb{U})} - z - B_1'(D_1 - z)^{-1}B_1 + \mathcal{E})_{xy}|, \end{aligned}$$

with $\mathcal{E}$ as in (3.91). For the sum, we have

(3.96)

$$\sum_{x,y\in\mathbb{T}\setminus\mathbb{U}} |(H^{(\mathbb{U})} - z - B_1'(D_1 - z)^{-1}B_1 + \mathcal{E})_{xy}|$$

$$\leqslant \sum_{x,y\in\mathbb{T}\setminus\mathbb{U}} H_{xy}^{(\mathbb{U})} + \sum_{\substack{i,j\in[1,\mu]\\ l_i,l_j\in\mathbb{T}_\ell\setminus\mathbb{U}}} (B_1')_{l_i a_i}|(D_1 - z)^{-1}_{a_i a_j}|(B_1)_{a_j l_j} + |z||\mathbb{T}\setminus\mathbb{U}| + |\mathbb{T}\setminus\mathbb{U}|^2 \max_{x,y\in\mathbb{T}\setminus\mathbb{U}} |\mathcal{E}_{xy}|$$

$$\leqslant \sum_{x,y\in\mathbb{T}\setminus\mathbb{U}} H_{xy}^{(\mathbb{U})} + \frac{1}{d-1} \sum_{i,j\in[\![1,\mu]\!]} |(D_1 - z)^{-1}_{a_i a_j}| + 2(d-1)^\ell |z| + 4(d-1)^{2\ell}\left(48(|z|+1)q^r\right),$$

where we used $|\mathbb{T}\setminus\mathbb{U}| \leqslant 2(d-1)^\ell$ and (3.89). By our assumption $\mathcal{G} \in \bar{\Omega}$, the subgraph $\mathcal{T}$ has excess at most ω. Therefore the total number of edges of $\mathcal{T}$ is bounded by $|\mathbb{T}| + \omega \leqslant 1 + d + d(d-1) + \cdots + d(d-1)^{\ell-1} + \omega \leqslant 2(d-1)^\ell$, and

(3.97)
$$\sum_{x,y\in\mathbb{T}\setminus\mathbb{U}} H_{xy}^{(\mathbb{U})} \leqslant \frac{2(d-1)^\ell}{\sqrt{d-1}}.$$

By the same argument as for (3.87), we get

(3.98)

$$\frac{1}{d-1}\sum_{i,j\in[\![1,\mu]\!]} |(D_1 - z)^{-1}_{a_i a_j}| = \frac{1}{d-1}\sum_{i\in[\![1,\mu]\!]} |(D_1 - z)^{-1}_{a_i a_i}| + \frac{1}{d-1}\sum_{i\neq j\in[\![1,\mu]\!]} |(D_1 - z)^{-1}_{a_i a_j}|$$

$$\leqslant \frac{3|m_{sc}|}{2}\left(\frac{\mu}{d-1} + \frac{(2\omega)^2}{d-1}\right) \leqslant \frac{3|m_{sc}|}{2}\left(2(d-1)^\ell + \frac{(2\omega)^2}{d-1}\right)$$

where we used $\mu \leqslant 2(d-1)^{\ell+1}$. By combining (3.96)–(3.98), we have

$$\sum_{x,y\in\mathbb{T}\setminus\mathbb{U}} |(H - z - B_1'(D_1 - z)^{-1}B_1 + \mathcal{E})_{xy}| \leqslant 2(d-1)^\ell|z| + 4(d-1)^{2\ell}\left(48(|z|+1)q^r\right) +$$

$$+ \frac{2(d-1)^\ell}{\sqrt{d-1}} + \frac{3|m_{sc}|}{2}\left(2(d-1)^\ell + \frac{(2\omega)^2}{d-1}\right) \leqslant 5(|z|+1)(d-1)^\ell.$$

Combining the above estimate with (3.95), and using $|zm_{sc}| \leqslant 2$,

$$|G_{ij}^{(\mathbb{T})} - G_{ij}^{(\mathbb{U})}| \leqslant 4|m_{sc}|^2 q^{2r} 5(|z|+1)(d-1)^\ell \leqslant \frac{20(|z|+1)|m_{sc}|}{\sqrt{d-1}}|m_{sc}|q^r \leqslant |m_{sc}|q^r/4.$$

This finishes the proof. $\qquad\square$

3.6. **Stability under switching.** We recall the S-cells and S′-cells from Definition 3.17, and the set of switching data $\mathsf{S}(\mathcal{G})$ from Section 3.3.2. The results of this section are the following stability estimates.

Proposition 3.26. *Let* $z \in \mathbb{C}_+$, $\mathcal{G} \in \bar{\Omega}$ *(as in Section 3.1.2) be a d-regular graph, and* $K \geqslant 2$ *be a constant such that, for all* $i, j \in [\![N]\!] \setminus \mathbb{T}$,

$$(3.99) \qquad \left| G_{ij}^{(\mathbb{T})} - P_{ij}(\mathcal{E}_r(i, j, \mathcal{G}^{(\mathbb{T})}), z) \right| \leqslant K |m_{sc}| q^r.$$

Then there exists an event $F(\mathcal{G}) \subset \mathsf{S}(\mathcal{G})$ *with* $\mathbb{P}_{\mathcal{G}}(F(\mathcal{G})) = 1 - \mathrm{o}(N^{-\omega+\delta})$, *explicitly defined in Section 3.6.1 below, such that for any* $\mathbf{S} \in F(\mathcal{G})$ *such that* $\tilde{\mathcal{G}} = T_{\mathbf{S}}(\mathcal{G}) \in \bar{\Omega}$ *the following hold:*

- *For* $i, j \in [\![N]\!] \setminus \mathbb{T}$,

$$(3.100) \qquad |\hat{G}_{ij}^{(\mathbb{T})} - P_{ij}(\mathcal{E}_r(i, j, \hat{\mathcal{G}}^{(\mathbb{T})}), z)| \leqslant 2K |m_{sc}| q^r.$$

- *For (i)* $i, j \in [\![N]\!] \setminus \mathbb{T}$ *in different* S*-cells, or (ii)* $i, j \in [\![N]\!] \setminus \mathbb{T}$ *such that* $j \in \mathbb{S}_t$ *and* $i \not\sim \mathbb{S}_t$ *for some* t,

$$(3.101) \qquad |\hat{G}_{ij}^{(\mathbb{T})}| \leqslant \frac{2M}{\sqrt{N\eta}}.$$

- *For* $i, j \in [\![N]\!] \setminus \mathbb{T}$,

$$(3.102) \qquad |\tilde{G}_{ij}^{(\mathbb{T})} - P_{ij}(\mathcal{E}_r(i, j, \tilde{\mathcal{G}}^{(\mathbb{T})}), z)| \leqslant 2^7 K^3 |m_{sc}| q^r.$$

For all estimates, we assume $\sqrt{d-1} \geqslant \max\{(\omega+1)^2 2^{2\omega+10}, 2^8(\omega+1)K\}$, $\omega' q^r \ll 1$ *and that* $\sqrt{N\eta} \geqslant M(d-1)^{\ell+1}$ *(where* M *is as in (3.60)).*

In particular, for any $\mathcal{G} \in \Omega(z, \ell)$, Proposition 3.21 implies that the assumptions of Proposition 3.26 are satisfied with $K = 2$. Thus Propositions 3.21 and 3.26 together

show that, for any graph $\mathcal{G} \in \Omega(z, \ell)$, with high probability under $\mathbf{S}$, the switched graph $\tilde{\mathcal{G}}$ belongs to $\Omega_1^+(z, \ell)$ (as in Section 3.1).

3.6.1. *Definition of the event $F(\mathcal{G})$.* We fix M and ω' by (3.60). We will prove Proposition 3.26 with the set $F(\mathcal{G}) \subset \mathsf{S}(\mathcal{G})$ defined by the following conditions on the switching data $\mathbf{S}$:

(i) At least $\mu - 3\omega$ edges are switchable, i.e. the event in the probability in (3.37) holds:

$$(3.103) \qquad |W_{\mathbf{S}}| \geqslant \mu - 3\omega.$$

(ii) All except for ω of the vertices $\{c_1, c_2, \ldots, c_\mu\}$ have radius-R tree neighborhoods in $\mathcal{G}^{(\mathbb{T})}$, i.e. (3.52) holds.

(iii) The vertices $\{a_1, \ldots, a_\mu, b_1, \ldots, b_\mu\}$ do not cluster in the sense of distance, i.e. (3.29)–(3.31) and (3.49)–(3.51) hold.

(iv) The vertices $\{b_1, b_2, \ldots, b_\mu\}$ do not cluster in the sense of the Green's function, i.e. (3.61)–(3.63) hold.

Then, for any $\mathcal{G} \in \Omega_1^+(z, \ell)$, we have

$$(3.104) \qquad \mathbb{P}_{\mathcal{G}}(F(\mathcal{G})) = 1 - o(N^{-\omega + \delta}).$$

Indeed, (i) follows from Proposition 3.13, (ii) follows from (3.52), (iii) follows from Propositions 3.10 and 3.15, and (iv) follows from Proposition 3.18.

3.6.2. *Proof of* (3.100). The proof of (3.100) follows the structure described below (3.71). Moreover, similarly to the proof of Proposition 3.21, we distinguish between vertices i, j that are close to the edges that get removed in going from $\mathcal{G}^{(\mathbb{T})}$ to $\hat{\mathcal{G}}^{(\mathbb{T})}$ and vertices that are far from these edges. We first focus on i, j that are close to those edges that get removed.

Localization. First, we replace $P_{ij}(\mathcal{E}_r(i,j,\mathcal{G}))$ by the vertex-independent Green's function P_{ij}, using Remark 2.8 with

$$(3.105) \qquad \mathcal{G}_0 = \mathcal{B}_{3r}(\{b_1, b_2, \ldots, b_\nu\}, \mathcal{G}^{(\mathbb{T})}), \quad \mathbb{X} = \mathbb{B}_{2r}(\{b_1, b_2, \ldots, b_\nu\}, \mathcal{G}^{(\mathbb{T})}).$$

Moreover, we define $\hat{\mathcal{G}}_0$ to be the graph obtained by removing the edges $\{b_i, c_i\}_{i \leqslant \nu}$ from $\mathcal{G}_0$. The deficit function of $\hat{\mathcal{G}}_0$ is defined to be the restriction of that of $\hat{\mathcal{G}}^{(\mathbb{T})}$. We abbreviate

$$\mathcal{G}_1 = \mathrm{TE}(\mathcal{G}_0), \quad P = G(\mathcal{G}_1), \quad \hat{\mathcal{G}}_1 = \mathrm{TE}(\hat{\mathcal{G}}_0), \quad \hat{P} = G(\hat{\mathcal{G}}_1).$$

Notice that $\hat{\mathcal{G}}_1$ is equivalently obtained by removing the edges $\{b_i, c_i\}_{i \leqslant \nu}$ from $\mathcal{G}_1$. The following properties of $\mathcal{G}_0$ follow from (3.50) and (3.51).

Claim 3.27. *Assume* (3.50) *and* (3.51). *Then each connected component of either $\mathcal{G}_0$ or $\hat{\mathcal{G}}_0$ contains at most 5ω elements from $\{a_1, \ldots, a_\mu, b_1, \ldots, b_\nu\}$. More precisely,*

$$(3.106) \qquad |\{i \in [\![1, \mu]\!] : a_i \in \mathbb{K}\}| \leqslant 3\omega, \quad |\{i \in [\![1, \nu]\!] : b_i \in \mathbb{K}\}| \leqslant 2\omega,$$

where $\mathbb{K}$ is the vertex set of any connected component of $\mathcal{G}_0$ or $\hat{\mathcal{G}}_0$.

Proof. The claim follows directly from (3.50) and (3.51) and the definitions of $\mathcal{G}_0$ and $\hat{\mathcal{G}}_0$. $\qquad \square$

Verification of assumptions in Proposition 2.7. Since both $\mathcal{G}_0$ and $\hat{\mathcal{G}}_0$ are subgraphs of $\mathcal{G} \in \bar{\Omega}$, their radius-$R$ neighborhoods have excess at most ω. Let $\mathbb{K}$ be the vertex set of any connected component of $\mathcal{G}_0$ or $\hat{\mathcal{G}}_0$. Since the deficit function of $\mathcal{G}_0$ (respectively $\hat{\mathcal{G}}_0$) is the restriction of that of $\mathcal{G}^{(\mathbb{T})}$ (respectively $\hat{\mathcal{G}}^{(\mathbb{T})}$), any of the vertices $a_i, b_i, c_i \in \mathbb{K}$ contributes 1 to the sum of the deficit function over $\mathbb{K}$. By Claim 3.27, the sums of the deficit functions over any of the connected components of $\mathcal{G}_0$ and $\hat{\mathcal{G}}_0$ are therefore bounded by $3\omega + 2 \times 2\omega \leqslant 8\omega$. Thus the assumptions of (2.14) are verified for both

$\mathcal{G}_0$ and $\hat{\mathcal{G}}_0$, and for any $i, j \in \mathbb{X}$,

$$(3.107) \qquad |P_{ij} - P_{ij}(\mathcal{E}_r((i, j, \mathcal{G}^{(\mathbb{T})}))| \leqslant 2^{2\omega+3}|m_{sc}|q^{r+1},$$

provided that $\sqrt{d-1} \geqslant 2^{\omega+2}$, and an analogous estimate holds for $\hat{P}$. Up to a small error, we can therefore use P instead of $P(\mathcal{E}_r((i, j, \mathcal{G}^{(\mathbb{T})}))$ and $\hat{P}$ instead of $P(\mathcal{E}_r((i, j, \hat{\mathcal{G}}^{(\mathbb{T})}))$.

Starting point. By the resolvent identity (B.1), we have:

$$(3.108) \qquad \hat{G}^{(\mathbb{T})} - G^{(\mathbb{T})} = G^{(\mathbb{T})}\Delta\hat{G}^{(\mathbb{T})},$$

$$(3.109) \qquad \hat{P} - P = P\Delta\hat{P},$$

where $\Delta = \sum_{k=1}^{\nu}(e_{b_k c_k} + e_{c_k b_k})/\sqrt{d-1}$. Taking the difference of (3.108) and (3.109), we obtain

$$(3.110)$$
$$\hat{G}^{(\mathbb{T})}_{ij} - \hat{P}_{ij} = (G^{(\mathbb{T})}_{ij} - P_{ij}) + \frac{1}{\sqrt{d-1}}\sum_{(x,y)\in\vec{E}}(G^{(\mathbb{T})}_{ix} - P_{ix})\hat{P}_{yj} + \frac{1}{\sqrt{d-1}}\sum_{(x,y)\in\vec{E}}G^{(\mathbb{T})}_{ix}(\hat{G}^{(\mathbb{T})}_{yj} - \hat{P}_{yj}),$$

where the summation is over the oriented edges

$$(3.111) \qquad (x, y) \in \vec{E} = \{(b_1, c_1), \ldots, (b_\nu, c_\nu), (c_1, b_1), \ldots, (c_\nu, b_\nu)\}.$$

We regard (3.110) as an equation for $\hat{G}^{(\mathbb{T})} - \hat{P}$, and will show that $\hat{G}^{(\mathbb{T})}_{ij} - \hat{P}_{ij}$ is small as a consequence of the smallness of $G^{(\mathbb{T})} - P$.

Green's function estimates. We first collect some estimates on Green's functions, used repeatedly:

$$(3.112) \qquad \begin{cases} |G^{(\mathbb{T})}_{ij}|, |P_{ij}|, |\hat{P}_{ij}| \leqslant 2|m_{sc}|, & \text{(all } i, j), \\[2mm] |G^{(\mathbb{T})}_{ij}| \leqslant K|m_{sc}|q^r, & (\text{dist}_{\mathcal{G}^{(\mathbb{T})}}(i, j) \geqslant 2r), \\[2mm] |G^{(\mathbb{T})}_{ib_k}|, |G^{(\mathbb{T})}_{ic_k}| \leqslant M/\sqrt{N\eta}, & (i, b_k \text{ are in different S-cells, or } i \nsim b_k). \end{cases}$$

The first estimate follows from (2.15), and (3.99); the second estimate follows from assumption (3.99), and $P_{ij}(\mathcal{E}_r(i,j,\mathcal{G}^{(\mathbb{T})})) = 0$; the last estimate holds by the definition of $\sim$ in Section 3.4.1.

Proof of (3.100) *for* $i, j \in \mathbb{X}$. By assumption and (3.107), the first term in (3.110) is bounded by $K|m_{sc}|q^r + 2^{2\omega+3}|m_{sc}|q^{r+1}$. For the second term on the right-hand side of (3.110), similarly $|G_{ix}^{(\mathbb{T})} - P_{ix}| \leqslant K|m_{sc}|q^r + 2^{2\omega+3}|m_{sc}|q^{r+1}$. Moreover, $\hat{P}_{yj} = 0$ if y and j are in different connected components of $\hat{\mathcal{G}}_0$ Thus by Claim 3.27, we have $\hat{P}_{yj} \neq 0$ for at most 4ω vertices $y \in \{b_i, c_i : i \in [\![1, \nu]\!]\}$, for which we use $|\hat{P}_{yj}| \leqslant 2|m_{sc}|$ by (3.112). Combining these bounds, the second term in (3.110) is bounded by

$$(3.113) \qquad \frac{1}{\sqrt{d-1}} \sum_{(x,y) \in \vec{E}} |G_{ix}^{(\mathbb{T})} - P_{ix}||\hat{P}_{yj}| \leqslant 8\omega(K + 2^{2\omega+3}q)|m_{sc}|q^{r+1}.$$

To estimate the last term in (3.110), we denote

$$\Gamma := \max_{i,j \in \mathbb{X}} |\hat{G}_{ij}^{(\mathbb{T})} - \hat{P}_{ij}|.$$

Noticing that $\mathbb{X} \subset \cup_{i=1}^{\kappa} \mathbb{S}_i$, we decompose the last sum over $\vec{E}$ in (3.110) according to the cases in (3.112) as

$$\sum_{\vec{E}} [\cdots] = \sum_{\vec{E}_1} [\cdots] + \sum_{\vec{E}_2} [\cdots] + \sum_{\vec{E}_3} [\cdots]$$

where here and below $[\cdots]$ abbreviates the terms in the last sum in (3.110) and

$$\vec{E}_1 = \{(b_k, c_k), (c_k, b_k) : i, b_k \text{ are in different S-cells}\},$$

$$\vec{E}_2 = \{(b_k, c_k), (c_k, b_k) : i, b_k \text{ are in the same S-cells, and } \text{dist}_{\mathcal{G}^{(\mathbb{T})}}(i, b_k) > 2r\},$$

$$\vec{E}_3 = \{(b_k, c_k), (c_k, b_k) : i, b_k \text{ are in the same S-cells, and } \text{dist}_{\mathcal{G}^{(\mathbb{T})}}(i, b_k) \leqslant 2r\}.$$

Notice that, for any $(x, y) \in \vec{E}$, we have $|(\hat{G}^{(\mathbb{T})} - \hat{P})_{yj}| \leqslant \Gamma$ by the definition of Γ.
For $(x, y) \in \vec{E}_1$, $|G_{ix}^{(\mathbb{T})}| \leqslant M/\sqrt{N\eta}$ by (3.112), and $|\vec{E}_1| \leqslant 2\nu \leqslant 4(d-1)^{\ell+1}$,

$$\sum_{\vec{E}_1}[\cdots] \leqslant \frac{\Gamma}{\sqrt{d-1}} \sum_{(x,y)\in\vec{E}_1} |G_{ix}^{(\mathbb{T})}| \leqslant \frac{4(d-1)^{\ell+1/2}M\Gamma}{\sqrt{N\eta}}.$$

For $(x, y) \in \vec{E}_2$, $|G_{ix}^{(\mathbb{T})}| \leqslant K|m_{sc}|q^r$ by (3.112), and $|\vec{E}_2| \leqslant 2\omega'$ by (3.62),

$$\sum_{\vec{E}_2}[\cdots] \leqslant \frac{\Gamma}{\sqrt{d-1}} \sum_{(x,y)\in\vec{E}_2} |G_{ix}^{(\mathbb{T})}| \leqslant \frac{2K\omega'|m_{sc}|q^r\Gamma}{\sqrt{d-1}}.$$

For $(x, y) \in \vec{E}_3$, $|G_{ix}^{(\mathbb{T})}| \leqslant 2|m_{sc}|$ by (3.112), and there are at most such 2ω terms, i.e.
$|\vec{E}_3| \leqslant 2\omega$, by (3.49),

$$\sum_{\vec{E}_3}[\cdots] \leqslant \frac{\Gamma}{\sqrt{d-1}} \sum_{(x,y)\in\vec{E}_3} |G_{ix}^{(\mathbb{T})}| \leqslant \frac{4\omega|m_{sc}|\Gamma}{\sqrt{d-1}}.$$

Combining the sums over $\vec{E}_1, \vec{E}_2, \vec{E}_3$, we get

$$\frac{1}{\sqrt{d-1}} \sum_{(x,y)\in\vec{E}} |G_{ix}^{(\mathbb{T})}||\hat{G}_{yj}^{(\mathbb{T})} - \hat{P}_{yj}| \leqslant \frac{\Gamma}{4},$$

provided that $\sqrt{d-1} \geqslant 20\omega$, $\omega'q^r \ll 1$ and $\sqrt{N\eta} \geqslant (d-1)^{\ell+1}M$. Thus (3.110) leads
to

$$|\hat{G}_{ij}^{(\mathbb{T})} - \hat{P}_{ij}| \leqslant (1 + 8\omega q)\left(K + 2^{2\omega+3}q\right)|m_{sc}|q^r + \Gamma/4.$$

Taking the supremum over $i, j \in \mathbb{X}$, we obtain

$$\Gamma \leqslant \frac{4}{3}(1 + 8\omega q)\left(K + 2^{2\omega+3}q\right)|m_{sc}|q^r.$$

From this estimate, and from (3.107) to estimate $\hat{P}_{ij} - P_{ij}(\mathcal{E}_r(i, j, \hat{\mathcal{G}}^{(\mathbb{T})}))$, we find

$$|\hat{G}_{ij}^{(\mathbb{T})} - P_{ij}(\mathcal{E}_r(i, j, \hat{\mathcal{G}}^{(\mathbb{T})}))| \leqslant |\hat{G}_{ij}^{(\mathbb{T})} - \hat{P}_{ij}| + |\hat{P}_{ij} - P_{ij}(\mathcal{E}_r((i, j, \hat{\mathcal{G}}^{(\mathbb{T})}))|$$

$$(3.114) \qquad \leqslant \frac{4}{3}(1 + 8\omega q)\left(K + 2^{2\omega+3}q\right)|m_{sc}|q^r + 2^{2\omega+3}|m_{sc}|q^{r+1} \leqslant 2K|m_{sc}|q^r,$$

provided that $\sqrt{d-1} \geqslant 2^{2\omega+5}$. This concludes the proof of (3.100) for $i, j \in \mathbb{X}$. $\qquad\square$

Proof of (3.100) *in the remaining case.* In the remaining case at least one of i, j is not contained in $\mathbb{X}$; and by symmetry we can assume that $i \notin \mathbb{X}$. Then $\mathcal{E}_r(i, j, \mathcal{G}^{(\mathbb{T})}) = \mathcal{E}_r(i, j, \hat{\mathcal{G}}^{(\mathbb{T})})$ and the graphs on both sides of the equality also have the same deficit function. It therefore suffices to show that $|\hat{G}_{ij}^{(\mathbb{T})} - G_{ij}^{(\mathbb{T})}|$ is small. By the resolvent identity (3.108), we have

$$(3.115) \qquad |\hat{G}_{ij}^{(\mathbb{T})} - G_{ij}^{(\mathbb{T})}| \leqslant \frac{1}{\sqrt{d-1}} \sum_{(x,y) \in \vec{E}} |G_{ix}^{(\mathbb{T})}||\hat{G}_{yj}^{(\mathbb{T})}|.$$

Since $i \notin \mathbb{X}$, we have $\text{dist}_{\mathcal{G}^{(\mathbb{T})}}(i, \{b_k, c_k\}) \geqslant 2r$ and therefore, by (3.112),

$$(3.116) \qquad |G_{ix}^{(\mathbb{T})}| \leqslant K|m_{sc}|q^r \quad \text{for any } x \in \{b_i, c_i : i \in [\![1, \nu]\!]\}.$$

For the case that exactly one of i, j is in $\mathbb{X}$, i.e. $i \notin \mathbb{X}$ and $j \in \mathbb{X}$, we now decompose the set $\vec{E}$ defined in (3.111) as $\vec{E} = \vec{E}_1' \cup \vec{E}_2' \cup \vec{E}_3'$, where

$$\vec{E}_1' = \{(b_k, c_k), (c_k, b_k) : \text{dist}_{\mathcal{G}^{(\mathbb{T})}}(b_k, j) \leqslant 2r\},$$

$$(3.117) \qquad \vec{E}_2' = \{(b_k, c_k), (c_k, b_k) : i \sim b_k, \text{dist}_{\mathcal{G}^{(\mathbb{T})}}(b_k, j) > 2r\},$$

$$\vec{E}_3' = \{(b_k, c_k), (c_k, b_k) : i \not\sim b_k, \text{dist}_{\mathcal{G}^{(\mathbb{T})}}(b_k, j) > 2r\}.$$

For $(x, y) \in \vec{E}_1'$, since $y, j \in \mathbb{X}$, $|\hat{G}_{yj}^{(\mathbb{T})}| \leqslant |P_{yj}(\mathcal{E}_r((y, j, \hat{\mathcal{G}}^{(\mathbb{T})}))| + 2K|m_{sc}|q^r \leqslant 2|m_{sc}|$ by (3.114) and (2.15), and there are at most 2ω terms, i.e. $|\vec{E}_1'| \leqslant 2\omega$ by (3.49),

$$\sum_{\vec{E}_1'}[\cdots] \leqslant \frac{1}{\sqrt{d-1}} \sum_{(x,y) \in \vec{E}_1'} (K|m_{sc}|q^r)|\hat{G}_{yj}^{(\mathbb{T})}| \leqslant \frac{4K\omega|m_{sc}|q^r}{\sqrt{d-1}}.$$

where now $[\cdots]$ refers to the terms in the sum in (3.115). For $(x, y) \in \vec{E}_2'$, since $y, j \in \mathbb{X}$, $|\hat{G}_{yj}^{(\mathbb{T})}| \leqslant 2K|m_{sc}|q^r$ by (3.114), and $|\vec{E}_2'| \leqslant 2\omega'$ by (3.61),

$$\sum_{\vec{E}_2'}[\cdots] \leqslant \frac{1}{\sqrt{d-1}} \sum_{(x,y) \in \vec{E}_2'} (K|m_{sc}|q^r)|\hat{G}_{yj}^{(\mathbb{T})}| \leqslant \frac{2\omega'(K|m_{sc}|q^r)(2K|m_{sc}|q^r)}{\sqrt{d-1}}.$$

For $(x, y) \in \vec{E}_3'$, $|G_{ix}^{(\mathbb{T})}| \leqslant M/\sqrt{N\eta}$ by the definition of $\sim$, $|\hat{G}_{yj}^{(\mathbb{T})}| \leqslant 2K|m_{sc}|q^r$ by (3.114), and $|\vec{E}_3'| \leqslant 2\nu \leqslant 4(d-1)^{\ell+1}$,

$$\sum_{\vec{E}_3'}[\cdots] \leqslant \frac{1}{\sqrt{d-1}} \sum_{(x,y)\in\vec{E}_3'} \frac{M}{\sqrt{N\eta}}(2K|m_{sc}|q^r) \leqslant 8K(d-1)^{\ell+1}q^{r+1}\frac{M}{\sqrt{N\eta}}.$$

Combining the sums over $\vec{E}_1'$, $\vec{E}_2'$, $\vec{E}_3'$, from (3.115) we obtain

$$|\hat{G}_{ij}^{(\mathbb{T})} - G_{ij}^{(\mathbb{T})}| \leqslant K|m_{sc}|q^r,$$

provided that $\sqrt{d-1} \geqslant 20\omega$, $\omega'q^r \ll 1$ and $\sqrt{N\eta} \geqslant (d-1)^{\ell+1}M$. This concludes the proof of (3.100) for $i \notin \mathbb{X}$ and $j \in \mathbb{X}$,

(3.118)
$$|\hat{G}_{ij}^{(\mathbb{T})} - P_{ij}(\mathcal{E}_r(i, j, \hat{\mathcal{G}}^{(\mathbb{T})}))| = |\hat{G}_{ij}^{(\mathbb{T})} - P_{ij}(\mathcal{E}_r(i, j, \mathcal{G}^{(\mathbb{T})}))|$$

$$\leqslant |\hat{G}_{ij}^{(\mathbb{T})} - G_{ij}^{(\mathbb{T})}| + |G_{ij}^{(\mathbb{T})} - P_{ij}(\mathcal{E}_r((i, j, \mathcal{G}^{(\mathbb{T})}))| \leqslant 2K|m_{sc}|q^r.$$

For the case that $i, j \notin \mathbb{X}$, noticing that $\mathrm{dist}_{\mathcal{G}^{(\mathbb{T})}}(b_k, j) > 2r$, we decompose the set $\vec{E}$ as $\vec{E} = \vec{E}_2' \cup \vec{E}_3'$, where $\vec{E}_2'$ and $\vec{E}_3'$ are defined in (3.117). By (3.118), for any $(x, y) \in \vec{E}$, $P_{yj}(\mathcal{E}_r(y, j, \hat{\mathcal{G}}^{(\mathbb{T})})) = 0$ and thus $|\hat{G}^{(\mathbb{T})}|_{yj} \leqslant 2K|m_{sc}|q^r$. Then the same argument as above implies

$$|\hat{G}_{ij}^{(\mathbb{T})} - G_{ij}^{(\mathbb{T})}| \leqslant K|m_{sc}|q^r.$$

This finishes the proof of the stability of $\hat{G}^{(\mathbb{T})}$. $\qquad\square$

3.6.3. *Proof of* (3.101). We again follow the structure described below (3.71), except that no localization step is required to prove (3.101).

Starting point. Under both conditions given for (3.101), we have $|G_{ij}^{(\mathbb{T})}| \leqslant M/\sqrt{N\eta}$ by the definition of $\sim$ as in Section 3.4.1. By the resolvent identity (3.108), we therefore

have

$$(3.119) \qquad |\hat{G}_{ij}^{(\mathbb{T})}| \leqslant \frac{M}{\sqrt{N\eta}} + \frac{1}{\sqrt{d-1}} \sum_{(x,y)\in\vec{E}} |G_{ix}^{(\mathbb{T})}||\hat{G}_{yj}^{(\mathbb{T})}|,$$

where $\vec{E}$ is as in (3.111). Notice that if i, j are in different S-cells, then for any $(x, y) \in \vec{E}$, either i, x are in different S-cells, or y, j are in different S-cells. Similarly if $j \in \mathbb{S}_t$ and $i \nsim \mathbb{S}_t$ for some t, then for any $(x, y) \in \vec{E}$, either $|G_{ix}^{(\mathbb{T})}| \leqslant M/\sqrt{N\eta}$, or the vertices y, j are in different S-cells. The claim (3.100) follows by analyzing (3.119) as an inequality for these $\hat{G}_{ij}^{(\mathbb{T})}$.

Green's function estimates. We first collect some estimates on Green's functions of $G^{(\mathbb{T})}$ and $\hat{G}^{(\mathbb{T})}$, which are repeatedly used in the proof: for $(x, y) \in \vec{E}$ as in (3.111),

(3.120)

$$|G_{ix}^{(\mathbb{T})}| \leqslant \begin{cases} 2|m_{sc}|, & (\text{all } x), \\ K|m_{sc}|q^r, & (\text{dist}_{\mathcal{G}^{(\mathbb{T})}}(i, x) \geqslant 2r), \\ M/\sqrt{N\eta}, & (i, x \text{ are in different S-cells; or } i \nsim \text{ the S-cell containing } x). \end{cases}$$

(3.121)

$$|\hat{G}_{yj}^{(\mathbb{T})}| \leqslant \begin{cases} 2|m_{sc}|, & (\text{all } y), \\ 2K|m_{sc}|q^r, & (\text{dist}_{\hat{\mathcal{G}}^{(\mathbb{T})}}(y, j) \geqslant 2r), \end{cases}$$

These estimates follow from (3.99)–(3.100), together with (2.15) for the bound for all x, y, and with $P_{ix}(\mathcal{E}_r(i, x, \mathcal{G}^{(\mathbb{T})})) = 0$ for $\text{dist}_{\mathcal{G}^{(\mathbb{T})}}(i, x) \geqslant 2r$; and $P_{yj}(\mathcal{E}_r(y, j, \hat{\mathcal{G}}^{(\mathbb{T})})) = 0$ for $\text{dist}_{\hat{\mathcal{G}}^{(\mathbb{T})}}(y, j) \geqslant 2r$. The last bound in (3.120) holds by the definition of $\sim$.

Proof of (3.101), *case (i).* We verify (3.101) in the case that i, j are in different S-cells. Denote

$$\Gamma := \max_{t_1 \neq t_2} \max_{i \in \mathbb{S}_{t_1}, j \in \mathbb{S}_{t_2}} |\hat{G}_{ij}^{(\mathbb{T})}|,$$

and now abbreviate by $[\cdots]$ the terms in the sum in (3.119) including the $1/\sqrt{d-1}$ prefactor. We divide the set $\vec{E}$ according to their relations to the cells $\mathbb{S}_{t_1}$ and $\mathbb{S}_{t_2}$ as $\vec{E} = \vec{E}_1 \cup \vec{E}_2 \cup \vec{E}_3 \cup \vec{E}_4 \cup \vec{E}_5$, where

$$\vec{E}_1 = \{(x,y) \in \mathbb{S}_{t_1} : \mathrm{dist}_{\mathcal{G}^{(\mathbb{T})}}(i,x) \leqslant 2r\},$$

$$\vec{E}_2 = \{(x,y) \in \mathbb{S}_{t_1} : \mathrm{dist}_{\mathcal{G}^{(\mathbb{T})}}(i,x) > 2r\},$$

$$\vec{E}_3 = \{(x,y) \in \mathbb{S}_{t_2} : \mathrm{dist}_{\mathcal{G}^{(\mathbb{T})}}(y,j) \leqslant 2r\},$$

$$\vec{E}_4 = \{(x,y) \in \mathbb{S}_{t_2} : \mathrm{dist}_{\mathcal{G}^{(\mathbb{T})}}(y,j) > 2r\},$$

$$\vec{E}_5 = \{(x,y) \notin \mathbb{S}_{t_1} \cup \mathbb{S}_{t_2}\}.$$

For $(x,y) \in \vec{E}_1$, $|\vec{E}_1| \leqslant 2\omega$ from (3.49), i.e. $|\{k \in [\![1,\nu]\!] : \mathrm{dist}(i,b_k) < 2r\}| \leqslant \omega$, and $|\hat{G}_{yj}^{(\mathbb{T})}| \leqslant \Gamma$, by the definition of Γ. Thus, by (3.120),

$$\sum_{\vec{E}_1}[\cdots] \leqslant \frac{4\omega|m_{sc}|\Gamma}{\sqrt{d-1}}.$$

For $(x,y) \in \vec{E}_2$, $|\vec{E}_2| \leqslant 2\omega'$ from (3.62), i.e. $|\mathbb{S}_{t_1} \cap \{b_1,\ldots,b_\nu\}| \leqslant \omega'$. Thus, by (3.120),

$$\sum_{\vec{E}_2}[\cdots] \leqslant \frac{2K\omega'|m_{sc}|q^r\Gamma}{\sqrt{d-1}}.$$

For $(x,y) \in \vec{E}_3$, $|\vec{E}_3| \leqslant 2\omega$ from (3.49), and by (3.120)–(3.121),

$$\sum_{\vec{E}_3}[\cdots] \leqslant \frac{M}{\sqrt{N\eta}} \frac{4\omega|m_{sc}|}{\sqrt{d-1}}$$

For $(x,y) \in \vec{E}_4$, $|\vec{E}_4| \leqslant 2\omega'$ from (3.62), and $\mathrm{dist}_{\hat{\mathcal{G}}^{(\mathbb{T})}}(y,j) \geqslant \mathrm{dist}_{\mathcal{G}^{(\mathbb{T})}}(y,j) > 2r$. Thus, by (3.120)–(3.121),

$$\sum_{\vec{E}_4}[\cdots] \leqslant \frac{M}{\sqrt{N\eta}} \frac{4K\omega'|m_{sc}|q^r}{\sqrt{d-1}}.$$

Finally, for $(x, y) \in \vec{E}_5$, we use $|\vec{E}_5| \leqslant 2\nu \leqslant 4(d-1)^{\ell+1}$ and $|\hat{G}_{yj}^{(\mathbb{T})}| \leqslant \Gamma$ which holds by the definition of Γ. Thus, by (3.120),

$$\sum_{\vec{E}_5}[\cdots] \leqslant \frac{4(d-1)^{\ell+1}M\Gamma}{\sqrt{d-1}\sqrt{N\eta}}.$$

Combining the sums over $\vec{E}_1, \ldots, \vec{E}_5$ in (3.119) leads to

$$|\hat{G}_{ij}^{(\mathbb{T})}| \leqslant \frac{M}{\sqrt{N\eta}} + \frac{(4\omega|m_{sc}| + 2K\omega'|m_{sc}|q^r)\,\Gamma}{\sqrt{d-1}}$$
$$+ \frac{(4\omega|m_{sc}| + 4K\omega'|m_{sc}|q^r)}{\sqrt{d-1}}\frac{M}{\sqrt{N\eta}} + \frac{4M(d-1)^{\ell+1/2}\Gamma}{\sqrt{N\eta}}.$$

By taking the maximum over i, j as in the assumption and rearranging the inequality, we get

(3.122) $$\Gamma \leqslant 2M/\sqrt{N\eta},$$

provided that $\sqrt{d-1} \geqslant 20\omega$, $\omega'q^r \ll 1$ and $\sqrt{N\eta} \geqslant M(d-1)^{\ell+1}$. $\qquad\square$

Proof of (3.101), case (ii). For $j \in \mathbb{S}_t$ and $i \not\sim \mathbb{S}_t$, we now decompose the set $\vec{E}$ according to their relations to vertex i and the cell $\mathbb{S}_t$ as $\vec{E} = \vec{E}_1' \cup \vec{E}_2' \cup \vec{E}_3' \cup \vec{E}_4' \cup \vec{E}_5'$, with

$$\vec{E}_1' = \{(b_k, c_k), (c_k, b_k) : i \not\sim b_k, b_k \notin \mathbb{S}_t\},$$

$$\vec{E}_2' = \{(b_k, c_k), (c_k, b_k) : i \sim b_k, \mathrm{dist}_{\mathcal{G}^{(\mathbb{T})}}(i, b_k) \leqslant 2r, b_k \notin \mathbb{S}_t\},$$

$$\vec{E}_3' = \{(b_k, c_k), (c_k, b_k) : i \sim b_k, \mathrm{dist}_{\mathcal{G}^{(\mathbb{T})}}(i, b_k) > 2r, b_k \notin \mathbb{S}_t\},$$

$$\vec{E}_4' = \{(b_k, c_k), (c_k, b_k) : b_k \in \mathbb{S}_t, \mathrm{dist}_{\mathcal{G}^{(\mathbb{T})}}(b_k, j) \leqslant 2r\},$$

$$\vec{E}_5' = \{(b_k, c_k), (c_k, b_k) : b_k \in \mathbb{S}_t, \mathrm{dist}_{\mathcal{G}^{(\mathbb{T})}}(b_k, j) > 2r\}.$$

For $(x, y) \in \vec{E}_1'$, $|\vec{E}_1'| \leqslant 2\nu \leqslant 4(d-1)^{\ell+1}$, and by (3.122), $|\hat{G}_{yj}^{(\mathbb{T})}| \leqslant 2M/\sqrt{N\eta}$, since y, j are in different S-cells. Thus, combining with (3.120),

$$\sum_{\vec{E}_1'}[\cdots] \leqslant 4(d-1)^{\ell+1}\frac{2M^2}{N\eta}\frac{1}{\sqrt{d-1}}.$$

For $(x, y) \in \vec{E}_2'$, y, j are in different S-cells. We have: $|\vec{E}_2'| \leqslant 2\omega$ from (3.49), i.e. $|\{k \in [\![1, \nu]\!] : \mathrm{dist}(i, b_k) \leqslant 2r\}| \leqslant \omega$, and $|\hat{G}_{yj}^{(\mathbb{T})}| \leqslant 2M/\sqrt{N\eta}$ by (3.122). Thus, by (3.120),

$$\sum_{\vec{E}_2'}[\cdots] \leqslant \frac{8\omega|m_{sc}|M}{\sqrt{d-1}\sqrt{N\eta}}.$$

For $(x, y) \in \vec{E}_3'$, y, j are in different S-cells. We have: $|\vec{E}_3'| \leqslant 2\omega'$ from (3.61), and $|\hat{G}_{yj}^{(\mathbb{T})}| \leqslant 2M/\sqrt{N\eta}$ by (3.122). Thus, by (3.120),

$$\sum_{\vec{E}_3'}[\cdots] \leqslant \frac{4K\omega'q^{r+1}M}{\sqrt{N\eta}}.$$

For $(x, y) \in \vec{E}_4'$, $|\vec{E}_4'| \leqslant 2\omega$ from (3.49), and (3.120)–(3.121),

$$\sum_{\vec{E}_4'}[\cdots] \leqslant \frac{4\omega|m_{sc}|M}{\sqrt{d-1}\sqrt{N\eta}}.$$

For $(x, y) \in \vec{E}_5'$, $|\vec{E}_5'| \leqslant 2\omega'$ from (3.62), and $\mathrm{dist}_{\hat{\mathcal{G}}^{(\mathbb{T})}}(y, j) \geqslant \mathrm{dist}_{\mathcal{G}^{(\mathbb{T})}}(y, j) \geqslant 2r$, since in the graph $\mathcal{G}^{(\mathbb{T})}$, b_k and c_k are adjacent. Thus, combining with (3.120)–(3.121),

$$\sum_{\vec{E}_5'}[\cdots] \leqslant \frac{4K\omega'q^{r+1}M}{\sqrt{N\eta}}.$$

Therefore, (3.119) can be bounded by

$$|\hat{G}_{ij}^{(\mathbb{T})}| \leqslant \frac{M}{\sqrt{N\eta}} + \left(\frac{12\omega|m_{sc}|M}{\sqrt{d-1}\sqrt{N\eta}} + \frac{8K\omega'q^{r+1}M}{\sqrt{N\eta}} + \frac{8M^2(d-1)^{\ell+1/2}}{N\eta}\right) \leqslant \frac{2M}{\sqrt{N\eta}},$$

given that $\sqrt{d-1} \geqslant 20\omega$, $\omega'q^r \ll 1$ and $\sqrt{N\eta} \geqslant M(d-1)^{\ell+1}$. $\qquad\square$

3.6.4. *Proof of* (3.102). As in previous arguments, we follow the structure described below (3.71).

Localization. The switching vertices that are not on the boundary of $\mathbb{T}$ *after* switching are given by $\{a_1, \ldots, a_\mu, b_1, \ldots, b_\nu\}$. From Section 3.4.1, we recall the partition $\{\mathbb{I}_1, \mathbb{I}_2, \ldots, \mathbb{I}_\kappa\}$ of this set. (Thus the $\mathbb{I}_j$ are the connected components of the Green's function graph $\mathcal{R}$, with all connected components containing any of the vertices a_i joined to $\mathbb{I}_1$.) The close vertices $\mathbb{X}_1 \cup \mathbb{X}_2$ and the larger subgraph $\mathcal{G}_0$ are defined by (3.123)

$$\mathcal{G}_0 := \mathcal{B}_{3r}(\{a_1, \ldots, a_\mu, b_1, \ldots, b_\nu\}, \mathcal{G}^{(\mathbb{T})}), \quad \mathbb{X}_1 := \mathbb{B}_{2r}(\mathbb{I}_1, \mathcal{G}^{(\mathbb{T})}), \quad \mathbb{X}_2 := \mathbb{B}_{2r}(\mathbb{I}_2 \cup \cdots \cup \mathbb{I}_\kappa, \mathcal{G}^{(\mathbb{T})}).$$

By our construction of S-cells and S′-cells, it follows that $\mathbb{X}_1 = \mathbb{S}_1 = \mathbb{S}_1'$ and $\mathbb{X}_2 = \cup_{i=2}^\kappa \mathbb{S}_i = \cup_{i=2}^{\kappa'} \mathbb{S}_i'$. By our conventions, the deficit function of the graph $\mathcal{G}_0$ is the restriction of that on $\mathcal{G}^{(\mathbb{T})}$. We define the graph $\hat{\mathcal{G}}_0$ by removing edges $\{b_i, c_i\}_{i \leqslant \nu}$ from $\mathcal{G}_0$ and $\tilde{\mathcal{G}}_0$ by adding edges $\{a_i, b_i\}_{i \leqslant \nu}$ to $\hat{\mathcal{G}}_0$. The graphs $\hat{\mathcal{G}}_0$ and $\tilde{\mathcal{G}}_0$ are given the restricted deficit functions from $\hat{\mathcal{G}}^{(\mathbb{T})}$ and $\tilde{\mathcal{G}}^{(\mathbb{T})}$ respectively. We abbreviate

$$\mathcal{G}_1 = \mathrm{TE}(\mathcal{G}_0), \quad \hat{\mathcal{G}}_1 = \mathrm{TE}(\hat{\mathcal{G}}_0), \quad \hat{P} = G(\hat{\mathcal{G}}_1), \quad \tilde{\mathcal{G}}_1 = \mathrm{TE}(\tilde{\mathcal{G}}_0), \quad \tilde{P} = G(\tilde{\mathcal{G}}_1).$$

Notice that the graph $\hat{\mathcal{G}}_1$ is obtained by removing the edges $\{b_i, c_i\}_{i \leqslant \nu}$ from $\mathcal{G}_1$, and that the graph $\tilde{\mathcal{G}}_1$ is obtained by adding the edges $\{b_i, a_i\}_{i \leqslant \nu}$ to $\hat{\mathcal{G}}_1$. We use the following fact throughout this section.

Claim 3.28. *If* (3.50) *and* (3.51) *hold, then each connected component of* $\tilde{\mathcal{G}}_0$ *contains at most* 10ω *elements in* $\{a_1, \ldots, a_\mu, b_1, \ldots, b_\nu\}$, *i.e.*

(3.124) $$|\{i \in [\![1, \mu]\!] : a_i \in \mathbb{K}\}| + |\{i \in [\![1, \nu]\!] : b_i \in \mathbb{K}\}| \leqslant 10\omega,$$

where $\mathbb{K}$ *is the vertex set of any connected component of* $\tilde{\mathcal{G}}_0$.

Proof. (3.124) is a consequence of Propositions 3.15 and 3.16. More precisely, if $a_i \in \mathbb{K}$ or $b_i \in \mathbb{K}$ for some $i \in [\![1, \mu]\!] \backslash (\mathsf{B}_a \cup \mathsf{B}_b)$, then $\mathbb{K}$ is disjoint from $\{a_1, a_2, \ldots, a_\mu, b_1, b_2, \ldots, b_\nu\} \backslash$

$\{a_i, b_i\}$. Therefore $|\{i \in [\![1, \mu]\!] : a_i \in \mathbb{K}\}| + |\{i \in [\![1, \nu]\!] : b_i \in \mathbb{K}\}| \leqslant 2|\mathsf{B}_a \cup \mathsf{B}_b| \leqslant 10\omega$. $\qquad\square$

Verification of assumptions in Proposition 2.7. By assumption $\tilde{\mathcal{G}} = T_{\mathbf{S}}(\mathcal{G}) \in \bar{\Omega}$. Since $\hat{\mathcal{G}}_0$ and $\tilde{\mathcal{G}}_0$ can be viewed as subgraphs of $\mathcal{G}$ and $\tilde{\mathcal{G}}$ respectively, the radius-R neighborhoods of them have excess at most ω. Moreover, the same argument as in Section 3.6.2 implies that the sum of the deficit functions on each connected component of $\hat{\mathcal{G}}_0$ and that of $\tilde{\mathcal{G}}_0$ are bounded by 8ω. Therefore the assumptions for (2.14) are verified for both graphs $\hat{\mathcal{G}}_0$ and $\tilde{\mathcal{G}}_0$. Thus (2.12)–(2.14) hold for $\hat{P}$ and $\tilde{P}$, and as in (3.107), we can use $\hat{P}$ instead of $P(\mathcal{E}_r(i, j, \hat{\mathcal{G}}^{(\mathbb{T})}))$ and $\tilde{P}$ instead of $P(\mathcal{E}_r(i, j, \tilde{\mathcal{G}}^{(\mathbb{T})}))$.

Starting point. The proof is similar to that of (3.100). By the resolvent identity (B.1), we have

(3.125) $$\tilde{G}^{(\mathbb{T})} - \hat{G}^{(\mathbb{T})} = \hat{G}^{(\mathbb{T})}\Delta\tilde{G}^{(\mathbb{T})},$$

(3.126) $$\tilde{P} - \hat{P} = \hat{P}\Delta\tilde{P}.$$

where $\Delta = \sum_{k=1}^{\nu}(e_{b_k a_k} + e_{a_k b_k})/\sqrt{d-1}$. Taking difference of (3.125) and (3.126), we have

(3.127)
$$\tilde{G}_{ij}^{(\mathbb{T})} - \tilde{P}_{ij} = (\hat{G}_{ij}^{(\mathbb{T})} - \hat{P}_{ij}) + \frac{1}{\sqrt{d-1}}\sum_{(x,y)\in\vec{E}}(\hat{G}_{ix}^{(\mathbb{T})} - \hat{P}_{ix})\tilde{P}_{yj} + \frac{1}{\sqrt{d-1}}\sum_{(x,y)\in\vec{E}}\hat{G}_{ix}^{(\mathbb{T})}(\tilde{G}_{yj}^{(\mathbb{T})} - \tilde{P}_{yj}),$$

where the sums are over the ordered pairs

(3.128) $$(x, y) \in \vec{E} = \{(a_1, b_1), \ldots, (a_\nu, b_\nu), (b_1, a_1), \ldots, (b_\nu, a_\nu)\}.$$

We regard (3.127) as an equation for $\tilde{G}^{(\mathbb{T})} - \tilde{P}$, and will show that $\tilde{G}^{(\mathbb{T})} - \tilde{P}$ is small, using that $\hat{G}^{(\mathbb{T})} - \hat{P}$ is small by (3.100).

Green's function estimates. We collect some estimates on Green's function, which are repeatedly used in the proof:

$$
(3.129) \quad
\begin{cases}
|\hat{G}_{ij}^{(\mathbb{T})}|, |\hat{P}_{ij}|, |\tilde{P}_{ij}| \leqslant 2|m_{sc}|, & \text{(all } i, j), \\
|\hat{G}_{ij}^{(\mathbb{T})}| \leqslant 2K|m_{sc}|q^r, & (\text{dist}_{\mathcal{G}^{(\mathbb{T})}}(i,j) \geqslant 2r), \\
|\hat{G}_{ib_k}^{(\mathbb{T})}|, |\hat{G}_{ic_k}^{(\mathbb{T})}| \leqslant 2M/\sqrt{N\eta}, & (i, b_k \text{ are in different S-cells; or } i \nsim b_k)
\end{cases}
$$

The first estimate follows from (2.15) and (3.100); the second estimate follows from $P_{ij}(\mathcal{E}_r(i,j,\hat{\mathcal{G}}^{(\mathbb{T})})) = 0$ and (3.100); the last estimate is from (3.101).

Proof of (3.102) *for* $i, j \in \mathbb{X}_1 \cup \mathbb{X}_2$. For the second term on the right-hand side of (3.127), it follows from (3.100) that $|\hat{G}_{ix}^{(\mathbb{T})} - \hat{P}_{ix}| \leqslant 2K|m_{sc}|q^r + 2^{2\omega+3}|m_{sc}|q^{r+1}$. Moreover, again $\tilde{P}_{yj} = 0$ if y and j are in different connected components of $\tilde{\mathcal{G}}_0$. Thus, by Claim 3.28, we have $\tilde{P}_{yj} \neq 0$ for at most 10ω vertices $y \in \{a_i : i \in [\![1, \mu]\!]\} \cup \{b_i : i \in [\![1, \nu]\!]\}$, and for these we again have $|\tilde{P}_{yj}| \leqslant 2|m_{sc}|$ by (3.129). Altogether, the second term on the right-hand of (3.127) is bounded by

$$
\frac{1}{\sqrt{d-1}} \sum_{(x,y) \in \vec{E}} |\hat{G}_{ix}^{(\mathbb{T})} - \hat{P}_{ix}||\tilde{P}_{yj}| \leqslant 20\omega(2K + 2^{2\omega+3}q)|m_{sc}|q^{r+1}.
$$

To estimate the last term in (3.127), we denote

$$
\Gamma_1 := \max_{i,j \in \mathbb{X}_1} |\tilde{P}_{ij} - \tilde{G}_{ij}^{(\mathbb{T})}|, \qquad \Gamma_2 := \max_{i \in \mathbb{X}_2, j \in \mathbb{X}_1 \cup \mathbb{X}_2} |\tilde{P}_{ij} - \tilde{G}_{ij}^{(\mathbb{T})}|.
$$

Our goal is to prove that

$$
(3.130) \qquad\qquad\qquad \Gamma_1, \Gamma_2 \leqslant 24K^2|m_{sc}|q^r.
$$

In the following, we first derive an estimate for Γ_2. We assume that $i \in \mathbb{S}_t$ for some $t \neq 1$, and $j \in \mathbb{X}_1 \cup \mathbb{X}_2$. We decompose the set $\vec{E}$ (as in (3.128)) according to their

relations to the S-cell $\mathbb{S}_t$: $\vec{E} = \vec{E}_1 \cup \vec{E}_2 \cup \vec{E}_3$, where

$$\vec{E}_1 = \{(x,y) : x \notin \mathbb{S}_t\},$$

$$\vec{E}_2 = \{(x,y) : x \in \mathbb{S}_t, \text{dist}_{\mathcal{G}^{(\mathbb{T})}}(i,x) < 2r\},$$

$$\vec{E}_3 = \{(x,y) : x \in \mathbb{S}_t, \text{dist}_{\mathcal{G}^{(\mathbb{T})}}(i,x) \geqslant 2r\}.$$

Notice that for any $(x,y) \in \vec{E}$, by the definition of Γ_1, Γ_2, we always have $|\tilde{P}_{yj} - \tilde{G}_{yj}^{(\mathbb{T})}| \leqslant$ $\max\{\Gamma_1, \Gamma_2\}$. For $(x,y) \in \vec{E}_1$, we have $|\vec{E}_1| \leqslant 2\nu \leqslant 4(d-1)^{\ell+1}$. Since i, x are in different S-cells, by (3.129),

$$\sum_{\vec{E}_1}[\cdots] \leqslant \frac{1}{\sqrt{d-1}} \sum_{(x,y)\in\vec{E}_1} |\hat{G}_{ix}^{(\mathbb{T})}| \max\{\Gamma_1, \Gamma_2\} \leqslant \frac{4(d-1)^{\ell+1}}{\sqrt{d-1}} \frac{2M}{\sqrt{N\eta}} \max\{\Gamma_1, \Gamma_2\}.$$

For $(x,y) \in \vec{E}_2$, we have $|\vec{E}_2| \leqslant 2\omega$ by (3.30) and (3.49). Thus, by (3.129),

$$\sum_{\vec{E}_2}[\cdots] \leqslant \frac{1}{\sqrt{d-1}} \sum_{(x,y)\in\vec{E}_2} |\hat{G}_{ix}^{(\mathbb{T})}| \max\{\Gamma_1, \Gamma_2\} \leqslant \frac{4\omega|m_{sc}|}{\sqrt{d-1}} \max\{\Gamma_1, \Gamma_2\}.$$

For $(x,y) \in \vec{E}_3$, we have $|\vec{E}_3| \leqslant \omega'$ from (3.62), and combined with (3.129),

$$\sum_{\vec{E}_3}[\cdots] \leqslant \frac{1}{\sqrt{d-1}} \sum_{(x,y)\in\vec{E}_3} |\hat{G}_{ix}^{(\mathbb{T})}| \max\{\Gamma_1, \Gamma_2\} \leqslant \frac{2K\omega'|m_{sc}|q^r}{\sqrt{d-1}} \max\{\Gamma_1, \Gamma_2\}.$$

Combining the sums over $\vec{E}_1, \vec{E}_2, \vec{E}_3$, for $i \in \mathbb{X}_2$ and $j \in \mathbb{X}_1 \cup \mathbb{X}_2$, (3.127) leads to

$$|\tilde{G}_{ij}^{(\mathbb{T})} - \tilde{P}_{ij}| \leqslant (20\omega q + 1)(2K + 2^{2\omega+3}q)|m_{sc}|q^r + \frac{8(\omega+1)}{\sqrt{d-1}} \max\{\Gamma_1, \Gamma_2\},$$

given $\sqrt{N\eta} \geqslant M(d-1)^{\ell+1}$, $\sqrt{d-1} \geqslant 20\omega$ and $\omega' q^r \ll 1$. Moreover, taking the maximum over all $i \in \mathbb{X}_2$ and $j \in \mathbb{X}_1 \cup \mathbb{X}_2$, we get

$$(3.131) \qquad \Gamma_2 \leqslant (20\omega q + 1)(2K + 2^{2\omega+3}q)|m_{sc}|q^r + \frac{8(\omega+1)}{\sqrt{d-1}} \max\{\Gamma_1, \Gamma_2\}.$$

Next, we estimate Γ_1. To this end, we decompose the set $\vec{E}$ (as in (3.128)) according to the cases in (3.129) as $\vec{E} = \vec{E}'_1 \cup \vec{E}'_2 \cup \vec{E}'_3 \cup \vec{E}'_4$, where

$$\vec{E}'_1 = \{(x, y) : x \in \mathbb{X}_1, \operatorname{dist}_{\mathcal{G}^{(\mathbb{T})}}(i, x) \leqslant 2r\},$$

$$\vec{E}'_2 = \{(x, y) : x, y \in \mathbb{X}_1, \operatorname{dist}_{\mathcal{G}^{(\mathbb{T})}}(i, x) > 2r\},$$

$$\vec{E}'_3 = \{(x, y) : x \in \mathbb{X}_1, y \in \mathbb{X}_2, \operatorname{dist}_{\mathcal{G}^{(\mathbb{T})}}(i, x) > 2r\},$$

$$\vec{E}'_4 = \{(x, y) : x \in \mathbb{X}_2\}.$$

For $(x, y) \in \vec{E}'_1$, $|\vec{E}'_1| \leqslant 2\omega$ from (3.30) and (3.49). $|\tilde{P}_{yj} - \tilde{G}^{(\mathbb{T})}_{yj}| \leqslant \max\{\Gamma_1, \Gamma_2\}$ by the definition of Γ_1, Γ_2. Therefore, by (3.129),

$$\sum_{\vec{E}'_1}[\cdots] \leqslant \frac{1}{\sqrt{d-1}} \sum_{(x,y)\in\vec{E}'_1} |\hat{G}^{(\mathbb{T})}_{ix}| \max\{\Gamma_1, \Gamma_2\} \leqslant \frac{4\omega|m_{sc}|}{\sqrt{d-1}} \max\{\Gamma_1, \Gamma_2\}.$$

For $(x, y) \in \vec{E}'_2$, $|\vec{E}'_2| \leqslant 2\omega'$ from (3.62), and $|\tilde{P}_{yj} - \tilde{G}^{(\mathbb{T})}_{yj}| \leqslant \Gamma_1$ from the definition of Γ_1. Thus, by (3.129),

$$\sum_{\vec{E}'_2}[\cdots] \leqslant \frac{1}{\sqrt{d-1}} \sum_{(x,y)\in\vec{E}'_2} |\hat{G}^{(\mathbb{T})}_{ix}| \Gamma_1 \leqslant 4K\omega' q^{r+1} \Gamma_1.$$

For $(x, y) \in \vec{E}'_3$, we have $|\vec{E}'_3| \leqslant 2\nu \leqslant 4(d-1)^{\ell+1}$, and $|\tilde{P}_{yj} - \tilde{G}^{(\mathbb{T})}_{yj}| \leqslant \Gamma_2$ from the definition of Γ_2. Thus, by (3.129),

$$\sum_{\vec{E}'_3}[\cdots] \leqslant \frac{1}{\sqrt{d-1}} \sum_{(x,y)\in\vec{E}'_3} |\hat{G}^{(\mathbb{T})}_{ix}| \Gamma_2 \leqslant 8K(d-1)^{\ell+1} q^{r+1} \Gamma_2.$$

For $(x, y) \in \vec{E}'_4$, we have $|\vec{E}'_4| \leqslant \nu \leqslant 2(d-1)^{\ell+1}$, and $|\tilde{P}_{yj} - \tilde{G}^{(\mathbb{T})}_{yj}| \leqslant \max\{\Gamma_1, \Gamma_2\}$ from the definition of Γ_1, Γ_2. Thus, by (3.129),

$$\sum_{\vec{E}'_4}[\cdots] \leqslant \frac{1}{\sqrt{d-1}} \sum_{(x,y)\in\vec{E}'_4} |\hat{G}^{(\mathbb{T})}_{ix}| (\Gamma_1 + \Gamma_2) \leqslant \frac{4(d-1)^{\ell+1} M}{\sqrt{d-1}\sqrt{N\eta}} \max\{\Gamma_1, \Gamma_2\}.$$

Using $r = 2\ell + 1$, and combining the above estimates in (3.127), we obtain that, for all $i, j \in \mathbb{X}_1$,

$$|\tilde{G}_{ij}^{(\mathbb{T})} - \tilde{P}_{ij}| \leqslant (20\omega q + 1)(2K + 2^{2\omega+3}q)|m_{sc}|q^r + \frac{8(\omega+1)}{\sqrt{d-1}}\Gamma_1 + \left(8K + \frac{4\omega+4}{\sqrt{d-1}}\right)\Gamma_2,$$

given $\sqrt{N\eta} \geqslant M(d-1)^{\ell+1}$ and $\omega' q^r \ll 1$. Taking the maximum over the left-hand side, we have

$$(3.132) \quad \Gamma_1 \leqslant (20\omega q + 1)(2K + 2^{2\omega+3}q)|m_{sc}|q^r + \frac{8(\omega+1)}{\sqrt{d-1}}\Gamma_1 + \left(8K + \frac{4\omega+4}{\sqrt{d-1}}\right)\Gamma_2.$$

Finally, the claim (3.130) follows by combining (3.131) and (3.132), provided that $\sqrt{d-1} \geqslant \max\{(\omega+1)^2 2^{2\omega+10}, 2^8(\omega+1)K\}$. Therefore for any $i, j \in \mathbb{X}_1 \cup \mathbb{X}_2$, we have

(3.133)
$$|\tilde{G}_{ij}^{(\mathbb{T})} - P_{ij}(\mathcal{E}_r(i, j, \tilde{G}^{(\mathbb{T})}))| \leqslant |(\tilde{G}^{(\mathbb{T})} - \tilde{P})_{ij}| + |\tilde{P}_{ij} - P_{ij}(\mathcal{E}_r(i, j, \tilde{G}^{(\mathbb{T})}))|$$

$$\leqslant 24K^2|m_{sc}|q^r + 2^{2\omega+3}|m_{sc}|q^{r+1} \leqslant (24K^2+1)|m_{sc}|q^r,$$

which implies the bound stated in (3.102). $\qquad\square$

Proof of (3.102) *for the remaining case.* For $i \notin \mathbb{X}_1 \cup \mathbb{X}_2$ and $j \in \mathbb{X}_1 \cup \mathbb{X}_2$, first note that $\mathcal{E}_r(i, j, \hat{G}^{(\mathbb{T})}) = \mathcal{E}_r(i, j, \tilde{G}^{(\mathbb{T})})$ and that both graphs have the same deficit function. To prove (3.102), we will show that $|\tilde{G}_{ij}^{(\mathbb{T})} - \hat{G}_{ij}^{(\mathbb{T})}|$ is small. To this end, we start from the resolvent identity (3.125), which states that

$$(3.134) \qquad\qquad |\tilde{G}_{ij}^{(\mathbb{T})} - \hat{G}_{ij}^{(\mathbb{T})}| \leqslant \frac{1}{\sqrt{d-1}} \sum_{(x,y)\in\vec{E}} |\hat{G}_{ix}^{(\mathbb{T})}||\tilde{G}_{yj}^{(\mathbb{T})}|.$$

By the definition of the sets $\mathbb{X}_1, \mathbb{X}_2$, for any $(x, y) \in \vec{E}$, we have $\text{dist}_{\mathcal{G}^{(\mathbb{T})}}(i, x) > 2r$ and $|\hat{G}_{ix}^{(\mathbb{T})}| \leqslant 2K|m_{sc}|q^r$ by (3.129). We simply decompose the set $\vec{E}$ (as in (3.128))

according to their distance to the vertex j as $\vec{E} = \vec{E}_1 \cup \vec{E}_2$, where

$$\vec{E}_1 = \{(x, y) : \text{dist}_{\tilde{\mathcal{G}}^{(\mathbb{T})}}(y, j) < 2r\},$$

$$\vec{E}_2 = \{(x, y) : \text{dist}_{\tilde{\mathcal{G}}^{(\mathbb{T})}}(y, j) \geqslant 2r\}.$$

For $(x, y) \in \vec{E}_1$, we have $|\vec{E}_1| \leqslant 10\omega$ by (3.124) in Claim 3.28. Moreover, $|\tilde{G}_{yj}^{(\mathbb{T})}| \leqslant |P_{yj}(\mathcal{E}_r(y, j, \tilde{G}^{(\mathbb{T})}))| + (24K^2 + 1)|m_{sc}|q^r \leqslant 2|m_{sc}|$ by (3.133). Thus, combining with (3.129), we have

$$\sum_{\vec{E}_1}[\cdots] \leqslant 40K\omega|m_{sc}|q^{r+1},$$

where here $[\cdots]$ denotes the terms in the sum in (3.134). For $(x, y) \in \vec{E}_2$, we have $|\vec{E}_2| \leqslant 2\nu \leqslant 2(d-1)^{\ell+1}$, and $|\tilde{G}_{yj}^{(\mathbb{T})}| \leqslant (24K^2 + 1)|m_{sc}|q^r$ by (3.133), since $P_{yj}(\mathcal{E}_r(y, j, \tilde{G}^{(\mathbb{T})})) = 0$. Thus, combining with (3.129),

$$\sum_{\vec{E}_2}[\cdots] \leqslant 2K(24K^2 + 1)|m_{sc}|^2 q^{2r} \frac{2(d-1)^{\ell+1}}{\sqrt{d-1}}.$$

Combining the sums over $\vec{E}_1, \vec{E}_2$, we get

$$|\tilde{G}_{ij}^{(\mathbb{T})} - \hat{G}_{ij}^{(\mathbb{T})}| \leqslant 40K\omega q^{r+1} + 2K(24K^2 + 1)|m_{sc}|^2 q^{2r} \frac{2(d-1)^{\ell+1}}{\sqrt{d-1}} \leqslant 100K^3|m_{sc}|q^r,$$

provided that $\sqrt{d-1} \geqslant 20\omega$. Similarly, in the case $i, j \notin \mathbb{X}_1 \cup \mathbb{X}_2$, we have

$$|\tilde{G}_{ij}^{(\mathbb{T})} - \hat{G}_{ij}^{(\mathbb{T})}| \leqslant \frac{1}{\sqrt{d-1}} \sum_{(x,y)\in\vec{E}} |\hat{G}_{ix}^{(\mathbb{T})}||\tilde{G}_{yj}^{(\mathbb{T})}| \leqslant 2K(24K^2+1)|m_{sc}|^2 q^{2r} \frac{2(d-1)^{\ell+1}}{\sqrt{d-1}} \leqslant 100K^3|m_{sc}|q^r.$$

Therefore, for $i \notin \mathbb{X}_1 \cup \mathbb{X}_2$ and $j \in \mathbb{X}_1 \cup \mathbb{X}_2$ or $i, j \notin \mathbb{X}_1 \cup \mathbb{X}_2$, we obtain

$$|\tilde{G}_{ij}^{(\mathbb{T})} - P_{ij}(\mathcal{E}_r(i, j, \tilde{G}^{(\mathbb{T})}))| \leqslant |\tilde{G}_{ij}^{(\mathbb{T})} - \hat{G}_{ij}^{(\mathbb{T})}| + |\hat{G}_{ij}^{(\mathbb{T})} - P_{ij}(\mathcal{E}_r(i, j, \hat{G}^{(\mathbb{T})}))|$$

$$\leqslant 100K^3|m_{sc}|q^r + 2K|m_{sc}|q^r \leqslant 2^7 K^3|m_{sc}|q^r.$$

This completes the proof of (3.102). $\qquad\square$

3.7. **Improved decay in the switched graph.** In the graph $\tilde{\mathcal{G}} = T_{\mathbf{S}}(\mathcal{G})$, the edge boundary $\partial_E \mathcal{T}$ and the vertex boundary $\partial \mathcal{T}$ of $\mathcal{T}$ are given by

$$(3.135) \qquad \partial_E \mathcal{T} = \{(l_1, \tilde{a}_1), (l_2, \tilde{a}_2), \ldots, (l_\mu, \tilde{a}_\mu)\}, \quad \mathbb{I} := \partial \mathcal{T} = \{\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_\mu\},$$

where the vertices $\tilde{a}_i = c_i$ with $i \in [\![1, \nu]\!]$ are those that get switched, and the vertices $\tilde{a}_i = a_i$ with $i \in [\![\nu + 1, \mu]\!]$ are those for which the switching does not take place. Here recall from Remark 3.14 that we assume without loss of generality that the index set of admissible switchings is $W_{\mathbf{S}} = [\![1, \nu]\!] \subset [\![1, \mu]\!]$.

The result of this section is the following proposition, showing that (i) between most vertices in $\mathbb{I}$ the Green's function is small; (ii) for any vertex not in $\mathbb{I}$, the Green's function between it and most vertices in $\mathbb{I}$ is also small. This decay asserted by the proposition is better than that between the boundary vertices of $\mathcal{T}$ which we assumed in the unswitched graph. This improvement is crucial for the subsequent sections, in particular for the derivation of the self-consistent equation.

Proposition 3.29. *Under the same assumptions as in Proposition 3.26, let $\mathbf{S} \in F(\mathcal{G})$ (as in Section 3.6.1) and assume that $\tilde{\mathcal{G}} = T_{\mathbf{S}}(\mathcal{G}) \in \bar{\Omega}$ (as in Section 3.1.2). Then there exists $J \subset [\![1, \nu]\!]$ with $|J| \geqslant \nu - \omega' - 6\omega$ such that, for any $k \in J$,*

$$(3.136) \qquad |\tilde{G}_{ic_k}^{(\mathbb{T})}| \leqslant 2^9 K^4 |m_{sc}| q^{2r+1} \quad \text{if } i = \tilde{a}_j \text{ for some } j \in [\![1, \mu]\!] \setminus J,$$

$$(3.137) \qquad |\tilde{G}_{ic_k}^{(\mathbb{T})}| \leqslant 2^{12} K^5 |m_{sc}| q^{3r+2} \quad \text{if } i = \tilde{a}_j \text{ for some } j \in J \setminus \{k\},$$

$$(3.138) \qquad |\tilde{G}_{ic_k}^{(\mathbb{T})}| \leqslant 2^{12} K^5 |m_{sc}| q^{2r+1} \quad \text{if } i \not\sim b_k \text{ and } \operatorname{dist}_{\tilde{\mathcal{G}}^{(\mathbb{T})}}(i, a_k) \geqslant 2r,$$

provided that $\sqrt{d-1} \geqslant \max\{(\omega+1)^2 2^{2\omega+10}, 2^8(\omega+1)K\}$, $\omega' q^r \ll 1$ and $\sqrt{N}\eta q^{3r+2} \geqslant M$.

The proposition uses the randomness of the resampling via the properties of the Green's function that are encoded by the S'-cells. Indeed, recall that if c_k was a random index, independent of $\tilde{\mathcal{G}}^{(\mathbb{T})}$ and i, then the size of the right-hand sides would

be of order $1/\sqrt{N\eta} \ll |m_{sc}|q^{3r+2}$ by the Ward identity (B.6). The remainder of this section is devoted to the proof of the proposition.

3.7.1. *Preliminaries.* To prove Proposition 3.29, we use the same setup as in the proof of (3.102). Thus, from (3.123) and the paragraph below, recall the sets $\mathbb{X}_1, \mathbb{X}_2$ and the graphs $\mathcal{G}_0, \hat{\mathcal{G}}_0, \tilde{\mathcal{G}}_0$, and that the set $\mathbb{X}_1 \cup \mathbb{X}_2$ is contained in $\mathbb{G}_0$ (the vertex set of $\mathcal{G}_0$). We also recall the S′-cells defined in Section 3.4.1.

We will prove Proposition 3.29 with the set $J \subset [\![1, \nu]\!]$ given by the set of indices $k \in [\![1, \nu]\!]$ such that the following conditions hold:

(i) $b_k, c_k \in \mathbb{X}_2$ (i.e. the S-cell containing b_k and c_k is not S_1);

(ii) $\mathcal{B}_R(c_k, \mathcal{G}^{(\mathbb{T})})$ is a tree;

(iii) the S′-cell $\mathbb{S}'$ containing b_k and c_k is not $\mathbb{S}'_1$ (as implied by (i)) and satisfies

$$(3.139) \quad \mathrm{dist}_{\tilde{\mathcal{G}}^{(\mathbb{T})}}(\mathbb{S}', \{a_m : m \in [\![1, \mu]\!] \setminus \{k\}\} \cup \{b_m, c_m : m \in [\![1, \nu]\!] \setminus \{k\}\}) > R/4.$$

By the assumption $\mathbf{S} \in F(\mathcal{G})$, and using the definition of $F(\mathcal{G})$ given in Section 3.6.1, note that (3.52) and (3.63) hold. (3.52) implies that condition (ii) in the definition of J is true for all $k \in [\![1, \nu]\!]$ with at most ω exceptions. (3.63) implies condition (i), and further that condition (iii) is true for all $k \in [\![1, \nu]\!]$ with at most $\omega' + 5\omega$ exceptions. It follows that

$$|J| \geqslant \nu - \omega' - 6\omega,$$

as asserted in the statement of Proposition 3.29. With this definition of J, to prove Proposition 3.29, we now follow the structure described below (3.71) (without the localization step, which is not required here).

Starting point. For the remainder of this section, we fix $k \in J$ and denote the S′-cell containing c_k by $\mathbb{S}'$. Notice that, by the definition of J, the S′-cell $\mathbb{S}'$ is not $\mathbb{S}'_1$, and that it is equal to the S-cell containing c_k. For any i arising in the statement of Proposition 3.29, we either have $i \in \mathbb{I}$, in which case i and c_k are in different S-cells

(by definition of J, the S-cell of c_k does not contain any $\tilde{a}_j$), or otherwise $i \nsim b_k$. Noticing that b_k and c_k are in the same S-cell, in both cases, the estimate (3.101) with $j = c_k$ holds. Therefore, since the graph $\tilde{\mathcal{G}}^{(\mathbb{T})}$ is given by adding the edges $\{a_i, b_i\}_{i \leqslant \nu}$ to $\hat{\mathcal{G}}^{(\mathbb{T})}$, by the resolvent formula (B.1), we have

(3.140)

$$|\tilde{G}^{(\mathbb{T})}_{ic_k}| = \left| \hat{G}^{(\mathbb{T})}_{ic_k} + \frac{1}{\sqrt{d-1}} \sum_{(x,y) \in \vec{E}} \hat{G}^{(\mathbb{T})}_{ix} \tilde{G}^{(\mathbb{T})}_{yc_k} \right| \leqslant \frac{2M}{\sqrt{N\eta}} + \frac{1}{\sqrt{d-1}} \sum_{(x,y) \in \vec{E}} |\hat{G}^{(\mathbb{T})}_{ix} \tilde{G}^{(\mathbb{T})}_{yc_k}|,$$

where the summation is over the ordered pairs

(3.141) $\qquad (x,y) \in \vec{E} = \{(a_1, b_1), \ldots, (a_\nu, b_\nu), (b_1, a_1), \ldots, (b_\nu, a_\nu)\}.$

By our assumption on η, the first term on the right-hand side of (3.140) is smaller than the right-hand sides of (3.136)–(3.138), so we only need to estimate the sum on the right-hand side of (3.140).

Green's function estimates. To estimate the sum on the right-hand side of (3.140), we use the following estimates on Green's functions, which hold for $(x,y) \in \vec{E}$:

(3.142)

$$|\hat{G}^{(\mathbb{T})}_{ix}| \leqslant \begin{cases} 2|m_{sc}| & \text{(all } x), \\ 2K|m_{sc}|q^r & (\text{dist}_{\hat{\mathcal{G}}^{(\mathbb{T})}}(i,x) \geqslant 2r), \\ 2M/\sqrt{N\eta} & (i \text{ and } x \text{ are in different S-cells, or } i \nsim \text{ the S-cell containing } x), \end{cases}$$

(3.143)

$$|\tilde{G}^{(\mathbb{T})}_{yc_k}| \leqslant \begin{cases} 2|m_{sc}| & \text{(all } y), \\ 2^7 K^3 |m_{sc}|q^r & (\text{dist}_{\tilde{\mathcal{G}}^{(\mathbb{T})}}(y, c_k) \geqslant 2r). \end{cases}$$

The last bound in (3.142) holds by (3.101). The remaining estimates follow from Propositions 3.26, together with (2.15) for the bound for all x, y; with $P_{ix}(\mathcal{E}_r(i, x, \hat{\mathcal{G}}^{(\mathbb{T})})) =$

0 for for the bound for $\mathrm{dist}_{\hat{\mathcal{G}}^{(\mathbb{T})}}(i, x) \geqslant 2r$; and with $P_{yc_k}(\mathcal{E}_r(y, c_k, \tilde{\mathcal{G}}^{(\mathbb{T})})) = 0$ for the bound for $\mathrm{dist}_{\tilde{\mathcal{G}}^{(\mathbb{T})}}(y, c_k) \geqslant 2r$.

Distance estimates. Since the estimates (3.142)–(3.143) depend on distances, we need some estimates on distances in the graphs $\hat{\mathcal{G}}^{(\mathbb{T})}$ and $\tilde{\mathcal{G}}^{(\mathbb{T})}$. These are summarized in the following lemma.

Lemma 3.30. *Let $k \in J$ and $\mathbb{S}'$ be the $\mathbb{S}'$-cell that contains c_k. Then the following estimates hold.*

(i) In the graph $\tilde{\mathcal{G}}^{(\mathbb{T})}$, the vertex c_k is far away from $\{a_1, \ldots, a_\mu, b_1, \ldots, b_\nu\}$:

$$(3.144) \qquad \mathrm{dist}_{\tilde{\mathcal{G}}^{(\mathbb{T})}}(c_k, \{a_1, \ldots, a_\mu, b_1, \ldots, b_\nu\}) > 2r.$$

(ii) If $\mathrm{dist}_{\tilde{\mathcal{G}}^{(\mathbb{T})}}(i, \mathbb{S}') > 2r$, then

$$(3.145) \qquad \mathrm{dist}_{\hat{\mathcal{G}}^{(\mathbb{T})}}(i, a_k) \geqslant \mathrm{dist}_{\tilde{\mathcal{G}}^{(\mathbb{T})}}(i, a_k) \geqslant 2r.$$

(iii) If $i \in \mathbb{X}_1$ and $\mathrm{dist}_{\tilde{\mathcal{G}}^{(\mathbb{T})}}(i, a_k) \geqslant 2r$, then

$$(3.146) \qquad \mathrm{dist}_{\tilde{\mathcal{G}}^{(\mathbb{T})}}(i, \mathbb{S}') > 2r.$$

Notice also that, by the definition of J, we have $\{m \in [\![1, \nu]\!] : b_m \in \mathbb{S}'\} = \{k\}$.

Proof. To prove (i), it follows from (3.139) from the definition of J that

$$\mathrm{dist}_{\tilde{\mathcal{G}}^{(\mathbb{T})}}(c_k, \{a_m : m \in [\![1, \mu]\!] \setminus \{k\} \cup \{b_m : m \in [\![1, \nu]\!] \setminus \{k\}\}\}) > R/4 > 2r.$$

It remains to prove $\mathrm{dist}_{\tilde{\mathcal{G}}^{(\mathbb{T})}}(c_k, \{a_k, b_k\}) > 2r$. Given any geodesic in $\tilde{\mathcal{G}}^{(\mathbb{T})}$ from c_k to $\{a_k, b_k\}$, we distinguish two cases. In the first case that the geodesic contains any of the edges $\{a_m, b_m\}_{m \leqslant \nu}$, the condition (3.139) which holds by the definition of J, implies that its length is larger than $2r$. In the second case that the geodesic contains none of the edges $\{a_m, b_m\}_{m \leqslant \nu}$, it a path on the graph $\hat{\mathcal{G}}^{(\mathbb{T})}$.

Therefore, to prove (i), it suffices to show that (3.144) holds with the graph $\tilde{\mathcal{G}}^{(\mathbb{T})}$ replaced by $\hat{\mathcal{G}}^{(\mathbb{T})}$. By the condition $b_k, c_k \in \mathbb{X}_2$, and since b_k, c_k are adjacent in $\mathcal{G}^{(\mathbb{T})}$, it follows from Lemma 3.19 that $\mathrm{dist}_{\mathcal{G}^{(\mathbb{T})}}(b_k, a_k) > 8r$, and therefore that

$$\mathrm{dist}_{\hat{\mathcal{G}}^{(\mathbb{T})}}(c_k, a_k) \geqslant \mathrm{dist}_{\mathcal{G}^{(\mathbb{T})}}(c_k, a_k) > 8r > 2r.$$

Moreover, since c_k has radius-R tree neighborhood in $\mathcal{G}^{(\mathbb{T})}$, and since in $\hat{\mathcal{G}}^{(\mathbb{T})}$ the edge $\{b_k, c_k\}$ is removed compared to $\mathcal{G}^{(\mathbb{T})}$, we have

$$\mathrm{dist}_{\hat{\mathcal{G}}^{(\mathbb{T})}}(b_k, c_k) > R > 2r.$$

This completes the proof of (3.144) with $\tilde{\mathcal{G}}^{(\mathbb{T})}$ replaced by $\hat{\mathcal{G}}^{(\mathbb{T})}$, and thus the proof of (i).

For (ii), since a_k and $b_k \in \mathbb{S}'$ are adjacent in the graph $\tilde{\mathcal{G}}^{(\mathbb{T})}$, we have

$$\mathrm{dist}_{\tilde{\mathcal{G}}^{(\mathbb{T})}}(i, a_k) \geqslant \mathrm{dist}_{\tilde{\mathcal{G}}^{(\mathbb{T})}}(i, \mathbb{S}') - 1 \geqslant 2r.$$

The first inequality in (3.145) is trivial since $\hat{\mathcal{G}}^{(\mathbb{T})} \subset \tilde{\mathcal{G}}^{(\mathbb{T})}$.

To prove (iii), note that any geodesic from i to $\mathbb{S}'$ in $\tilde{\mathcal{G}}^{(\mathbb{T})}$ either contains a_k, or does not contain the edge $\{a_k, b_k\}$. In the first case that the geodesic contains a_k, its length is at least $1 + \mathrm{dist}_{\tilde{\mathcal{G}}^{(\mathbb{T})}}(i, a_k) > 2r$, as desired. In the second case,

$$\mathrm{dist}_{\tilde{\mathcal{G}}^{(\mathbb{T})}}(i, \mathbb{S}') \geqslant \mathrm{dist}_{\tilde{\mathcal{G}}^{(\mathbb{T})} \setminus \{a_k, b_k\}}(\mathbb{X}_1 \cup \mathbb{X}_2 \setminus \mathbb{S}', \mathbb{S}') = \mathrm{dist}_{\mathcal{G}^{(\mathbb{T})}}(\mathbb{X}_1 \cup \mathbb{X}_2 \setminus \mathbb{S}', \mathbb{S}') > 4r,$$

where the first inequality holds since $i \in \mathbb{X}_1 \cup \mathbb{X}_2 \setminus \mathbb{S}'$, and the last inequality follows from the definition of the S-cells and Lemma 3.19. Recall that the graph $\tilde{\mathcal{G}}^{(\mathbb{T})}$ is obtained from $\mathcal{G}^{(\mathbb{T})}$ by adding the edges $\{a_m, b_m\}_{m \leqslant \nu}$ and removing the edges $\{b_m, c_m\}_{m \leqslant \nu}$. And by the definition of the set J, we know $\{b_k, c_k\} \subset \mathbb{S}'$ and $\{b_m, c_m : m \in [\![1, \nu]\!] \setminus \{k\}\} \subset \mathbb{X}_1 \cup \mathbb{X}_2 \setminus \mathbb{S}'$. Therefore, the graph $\tilde{\mathcal{G}}^{(\mathbb{T})} \setminus \{a_k, b_k\}$ and $\mathcal{G}^{(\mathbb{T})}$ are

different only on the subgraphs induced on $\mathbb{S}'$ and $\mathbb{X}_1 \cup \mathbb{X}_2 \setminus \mathbb{S}'$, and the equality in the above equation holds. $\qquad\square$

The proof of Proposition 3.29 essentially follows from the heuristic described in Remark **??**, which can be made rigorous by combining the estimates on the Green's function of (3.142)–(3.143) with those on the distances stated in Lemma 3.30. This requires a division into a number of cases and is done carefully below.

3.7.2. *Proof of* (3.136). Let

$$(3.147) \qquad \Gamma_1 := \max\left\{|\tilde{G}_{ic_k}^{(\mathbb{T})}| : i \in \mathbb{X}_1 \text{ such that } \mathrm{dist}_{\tilde{\mathcal{G}}^{(\mathbb{T})}}(i, \mathbb{S}') > 2r\right\},$$

$$(3.148) \qquad \Gamma_2 := \max\left\{|\tilde{G}_{ic_k}^{(\mathbb{T})}| : i \in \mathbb{X}_2 \text{ and } i \notin \mathbb{S}'\right\}.$$

Thus Γ_1 is the maximal size of the Green's function between c_k and vertices in $\mathbb{X}_1$ which is away from $\mathbb{S}'$, and Γ_2 is the maximal size of the Green's function between c_k and vertices in $\mathbb{X}_2$ which is in different $\mathbb{S}'$-cells from c_k.

Proposition 3.31.

$$(3.149) \qquad\qquad \max\{\Gamma_1, \Gamma_2\} \leqslant 2^9 K^4 |m_{sc}| q^{2r+1},$$

provided that $\sqrt{d-1} \geqslant \max\{(\omega+1)^2 2^{2\omega+10}, 2^8(\omega+1)K\}$, $\omega' q^r \ll 1$ *and* $\sqrt{N\eta} q^{2r+2} \geqslant M$.

Given Proposition 3.31, the claim (3.136) is an immediate consequence.

Proof of (3.136). It suffices to show that the left-hand side of (3.136) is bounded by $\max\{\Gamma_1, \Gamma_2\}$. First, if $i \in \mathbb{X}_2$, then $i = c_l$ for some $l \neq k$, and by the definition of J, then $c_l \notin \mathbb{S}'$. Thus the left-hand sides of (3.136) is bounded by Γ_2. Second, if $i \in \mathbb{X}_1$, then either $i = a_l$ or $i = c_l$ for some $l \neq k$. In either case, by the definition of J, $\mathrm{dist}_{\tilde{\mathcal{G}}^{(\mathbb{T})}}(i, \mathbb{S}') > R/4 - 2r > 2r$. Thus the left-hand side of (3.136) is bounded by Γ_1. $\qquad\square$

Proof of Proposition 3.31. We first derive a bound for Γ_1. Let i obey the conditions in the definition of Γ_1 in (3.147). We divide the sum over the set $\vec{E}$ in (3.140) according to the cases in (3.142)–(3.143) as $\vec{E} = \vec{E}_1 \cup \cdots \cup \vec{E}_5$, where

$$\vec{E}_1 = \{(a_k, b_k)\},$$

$$\vec{E}_2 = \{(b_k, a_k)\},$$

$$\vec{E}_3 = \{(b_l, a_l) : l \neq k, b_l \in \mathbb{X}_2\},$$

$$\vec{E}_4 = \{(a_l, b_l) : l \neq k, b_l \in \mathbb{X}_2\},$$

$$\vec{E}_5 = \{(a_l, b_l), (b_l, a_l) : l \neq k, b_l \in \mathbb{X}_1\}.$$

For $(a_k, b_k) \in \vec{E}_1$, we have $\text{dist}_{\hat{\mathcal{G}}^{(\mathbb{T})}}(i, a_k) \geq 2r$ by (3.145) and $\text{dist}_{\tilde{\mathcal{G}}^{(\mathbb{T})}}(b_k, c_k) \geq 2r$ by (3.144). Thus, by (3.142)–(3.143),

$$\sum_{\vec{E}_1}[\cdots] \leq \frac{(2K|m_{sc}|q^r)(2^7 K^3 |m_{sc}|q^r)}{\sqrt{d-1}}.$$

For $(b_k, a_k) \in \vec{E}_2$, we have $b_k \in \mathbb{X}_2$ and $i \in \mathbb{X}_1$, which implies i and b_k are in different S-cells. Thus, by (3.142)–(3.143),

$$\sum_{\vec{E}_2}[\cdots] \leq \frac{2M}{\sqrt{N\eta}} \frac{2|m_{sc}|}{\sqrt{d-1}} \leq \frac{4qM}{\sqrt{N\eta}}.$$

For $(b_l, a_l) \in \vec{E}_3$, we again have that i and b_l are in different S-cells (since $b_l \in \mathbb{X}_2$) and $\text{dist}_{\tilde{\mathcal{G}}^{(\mathbb{T})}}(a_l, c_k) \geq 2r$ by (3.144). Thus, by (3.142)–(3.143) and $|\vec{E}_3| \leq \mu \leq 2(d-1)^{\ell+1}$,

$$\sum_{\vec{E}_3}[\cdots] \leq \frac{M}{\sqrt{N\eta}} 2^9 K^3 (d-1)^{\ell+1} q^{r+1}.$$

For $(a_l, b_l) \in \vec{E}_4$, there are at most $\omega+1$ indices l such that $\text{dist}_{\mathcal{G}^{(\mathbb{T})}}(i, a_l) \leq \text{dist}_{\hat{\mathcal{G}}^{(\mathbb{T})}}(i, a_l) < 2r$ by (3.30), and at most $|\vec{E}_4| \leq \mu \leq 2(d-1)^{\ell+1}$ indices such that $\text{dist}_{\hat{\mathcal{G}}^{(\mathbb{T})}}(i, a_l) \geq 2r$. Moreover, we have $b_l \in \mathbb{X}_2$ and also $b_l \notin \mathbb{S}'$ by the definition of J. Thus, by (3.142)

and the definition of Γ_2,

$$\sum_{\vec{E}_4}[\cdots] \leqslant \left((\omega+1)\frac{2|m_{sc}|}{\sqrt{d-1}} + 2(d-1)^{\ell+1}\frac{2K|m_{sc}|q^r}{\sqrt{d-1}}\right)\Gamma_2.$$

For $(x,y) \in \vec{E}_5$, there are at most 10ω pairs $(x,y) \in \vec{E}_5$ such that $\text{dist}_{\tilde{\mathcal{G}}^{(\mathbb{T})}}(i,x) < 2r$ by (3.54) in Proposition 3.16, at most $2\omega'$ pairs such that $\text{dist}_{\tilde{\mathcal{G}}^{(\mathbb{T})}}(i,x) \geqslant 2r$ since $|\mathbb{X}_1 \cap \{b_1, \ldots, b_\nu\}| \leqslant \omega'$ by (3.62). Thus, by (3.142) and $|\tilde{G}_{yc_k}^{(\mathbb{T})}| \leqslant \Gamma_1$

$$\sum_{\vec{E}_5}[\cdots] \leqslant \left(10\omega\frac{2|m_{sc}|}{\sqrt{d-1}} + 2\omega'\frac{2K|m_{sc}|q^r}{\sqrt{d-1}}\right)\Gamma_1.$$

Combining the sums over $\vec{E}_1, \ldots \vec{E}_5$, and taking the maximum over i obeying the conditions in the definition of Γ_1 in (3.147), we get

(3.150)
$$\Gamma_1 \leqslant (2 + 4q + 2^9 K^3)\frac{M}{\sqrt{N\eta}} + 2^8 K^4|m_{sc}|q^{2r+1} + (20\omega q + 4K\omega' q^{r+1})\Gamma_1 + (2(\omega+1)q + 4K)\Gamma_2.$$

To bound Γ_2, let $i \in \mathbb{X}_2$ be as in the definition of Γ_2. Let $\mathbb{S}''$ be the $\mathbb{S}'$-cell containing i, and notice that $\mathbb{S}' \neq \mathbb{S}'', \mathbb{S}'_1$ from the definition of Γ_2. We now divide $\vec{E} = \vec{E}'_1 \cup \cdots \cup \vec{E}'_4$ where

(3.151)
$$\vec{E}'_1 = \{(x,y) : x \in X_1\},$$

(3.152)
$$\vec{E}'_2 = \{(b_l, a_l) : b_l \in \mathbb{S}'\} = \{(b_k, a_k)\},$$

(3.153)
$$\vec{E}'_3 = \{(b_l, a_l) : b_l \in \mathbb{S}''\},$$

(3.154)
$$\vec{E}'_4 = \{(b_l, a_l) : b_l \in \mathbb{X}_2 \setminus (\mathbb{S}' \cup \mathbb{S}'')\}.$$

For $(x,y) \in \vec{E}'_1$, i and x are in different S-cells (since $x \in \mathbb{X}_1$ and $i \in \mathbb{X}_2$) and $\text{dist}_{\tilde{\mathcal{G}}^{(\mathbb{T})}}(y, c_k) > 2r$ by (3.144). Since $|\vec{E}'_1| \leqslant 2\mu \leqslant 4(d-1)^{\ell+1}$

$$\sum_{\vec{E}'_1}[\cdots] \leqslant 4(d-1)^{\ell+1}\frac{2M}{\sqrt{N\eta}}\frac{2^7 K^3|m_{sc}|q^r}{\sqrt{d-1}}.$$

For $(b_k, a_k) \in \vec{E}_2'$, i and b_k are in different S-cells since $i \in \mathbb{S}''$ and $b_k \in \mathbb{S}'$ by assumption. Moreover, we have $\text{dist}_{\tilde{\mathcal{G}}^{(\mathbb{T})}}(a_k, c_k) > 2r$ by (3.144). Thus

$$\sum_{\vec{E}_2'}[\cdots] \leqslant \frac{2M}{\sqrt{N\eta}} \frac{2^7 K^3 |m_{sc}| q^r}{\sqrt{d-1}}.$$

For $(b_l, a_l) \in \vec{E}_3'$, there are at most 5ω indices l such that $\text{dist}_{\tilde{\mathcal{G}}^{(\mathbb{T})}}(i, b_l) < 2r$ by (3.54) in Proposition 3.16, and at most $|\vec{E}_3'| \leqslant |\{l \in [\![1, \nu]\!] : b_l \in \mathbb{S}''\}| \leqslant \omega' + 5\omega$ indices such that $\text{dist}_{\tilde{\mathcal{G}}^{(\mathbb{T})}}(i, b_l) \geqslant 2r$ by (3.63). Moreover, $|\tilde{G}_{a_l c_k}^{(\mathbb{T})}| \leqslant \Gamma_1$ (since $\text{dist}_{\tilde{\mathcal{G}}^{(\mathbb{T})}}(a_l, \mathbb{S}') > R/4 - 2r > 2r$ by the definition of J). Thus

$$\sum_{\vec{E}_3'}[\cdots] \leqslant \left(5\omega \frac{2|m_{sc}|}{\sqrt{d-1}} + (\omega' + 5\omega)\frac{2K|m_{sc}|q^r}{\sqrt{d-1}}\right)\Gamma_1.$$

For $(b_l, a_l) \in \vec{E}_4'$, i and b_l are in different S-cells; a_l and c_k are in different S-cells (since $a_l \in \mathbb{S}_1'$ and $c_k \in \mathbb{S}'$); there are at most $|\vec{E}_4'| \leqslant \mu \leqslant 2(d-1)^{\ell+1}$ terms. Thus

$$\sum_{\vec{E}_4'}[\cdots] \leqslant 2(d-1)^{\ell+1}\frac{2M}{\sqrt{N\eta}}\frac{\Gamma_1}{\sqrt{d-1}}.$$

Combining the sums over $\vec{E}_1', \ldots, \vec{E}_4'$, and taking the maximum over i obeying the conditions in the definition of Γ_2, we get

(3.155)
$$\Gamma_2 \leqslant \left(2 + 2^{10}K^3 + 2^8 K^3 q^{r+1}\right)\frac{M}{\sqrt{N\eta}} + \left(10\omega q + 2K(\omega' + 5\omega)q^{r+1} + \frac{4(d-1)^{\ell+1/2}M}{\sqrt{N\eta}}\right)\Gamma_1.$$

In summary, in (3.150) and (3.155), we have shown that

$$\Gamma_1 \leqslant \mathfrak{a} + \mathfrak{b}\Gamma_1 + \mathfrak{c}\Gamma_2, \qquad \Gamma_2 \leqslant \mathfrak{d} + \mathfrak{e}\Gamma_1,$$

where $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}, \mathfrak{e}$ are explicit constants given in (3.150) and (3.155). By plugging the second estimate into the first one, noticing $\mathfrak{b} + \mathfrak{c}\mathfrak{e} < 1$, and using the explicit values

of $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}, \mathfrak{e}$, it follows that

$$\Gamma_1 \leqslant (\mathfrak{a} + \mathfrak{c}\mathfrak{d})/(1 - (\mathfrak{b} + \mathfrak{c}\mathfrak{e})) \leqslant 2^9 K^4 |m_{sc}| q^{2r+1}, \qquad \Gamma_2 \leqslant \mathfrak{d} + \mathfrak{e}\Gamma_1 \leqslant 2^9 K^4 |m_{sc}| q^{2r+1},$$

provided that $\sqrt{d-1} \geqslant \max\{(\omega+1)^2 2^{2\omega+10}, 2^8(\omega+1)K\}$, $\omega' q^r \ll 1$ and $\sqrt{N}\eta q^{2r+2} \geqslant M$. $\qquad\square$

3.7.3. *Proofs of* (3.138) *and* (3.137).

Proof of (3.138). To bound the left-hand side of (3.138), consider first the case that $i \in \mathbb{X}_1 \cup \mathbb{X}_2$: (i) if $i \in \mathbb{X}_1$ and $\mathrm{dist}_{\tilde{\mathcal{G}}^{(\mathbb{T})}}(i, a_k) \geqslant 2r$, it follows by (3.146) that $\mathrm{dist}_{\tilde{\mathcal{G}}^{(\mathbb{T})}}(i, \mathbb{S}') > 2r$. Thus the left-hand side of (3.138) is bounded by Γ_1; (ii) if $i \in \mathbb{X}_2$ and $i \not\sim b_k$, then $i \notin \mathbb{S}'$, and the left-hand side of (3.138) is bounded by Γ_2. Therefore (3.138) follows from Proposition 3.31.

For the remaining case $i \notin \mathbb{X}_1 \cup \mathbb{X}_2$ and $i \not\sim b_k$, we bound the sum over $\vec{E}$ in (3.140). By the definition of $\mathbb{X}_1$ and $\mathbb{X}_2$, $\mathrm{dist}_{\mathcal{G}^{(\mathbb{T})}}(i, \{a_1, \ldots, a_\mu, b_1, \ldots, b_\nu\}) > 2r$, and therefore also in $\hat{\mathcal{G}}^{(\mathbb{T})} \subset \mathcal{G}^{(\mathbb{T})}$. Thus (3.142) implies $|\hat{G}_{ix}^{(\mathbb{T})}| \leqslant 2K|m_{sc}|q^r$ for all $x \in \{a_1, \ldots, a_\nu, b_1, \ldots, b_\nu\}$.

For $(x, y) = (a_k, b_k), (b_k, a_k)$, we have $\mathrm{dist}_{\tilde{\mathcal{G}}^{(\mathbb{T})}}(y, c_k) > 2r$ by (3.144), and thus $|\tilde{G}_{yc_k}^{(\mathbb{T})}| \leqslant 2^7 K^3 |m_{sc}| q^r$ by (3.143). The remaining $y \neq a_k, b_k$ satisfy either the condition in (3.147) or in (3.148). Therefore $|\tilde{G}_{yc_k}^{(\mathbb{T})}| \leqslant \max\{\Gamma_1, \Gamma_2\} \leqslant \Gamma_1$, and there are at most $2\mu \leqslant 2d(d-1)^\ell$ such terms.

In summary, we have shown

$$|\tilde{G}_{ic_k}^{(\mathbb{T})}| \leqslant \frac{2M}{\sqrt{N}\eta} + 2^9 K^4 |m_{sc}| q^{2r+1} + (2Kq^{r+1})(2d(d-1)^\ell)\Gamma_1 \leqslant 2^{12} K^5 |m_{sc}| q^{2r+1},$$

provided that $\sqrt{N}\eta \geqslant M q^{-2r-2}$, where we used $r = 2\ell + 1$. $\qquad\square$

Proof of (3.137). It remains to estimate $\tilde{G}_{c_j c_k}^{(\mathbb{T})}$ for $j \in J \setminus \{k\}$. As previously, we denote by $\mathbb{S}'$ the $\mathbb{S}'$-cell containing c_k, and now denote by $\mathbb{S}''$ the $\mathbb{S}'$-cell containing c_j. The estimates in Lemma 3.30 on distances from c_k also apply with c_k replaced by c_j.

Similarly to the bound of Γ_2, we use formula (3.140) and devide $\vec{E}$ as $\vec{E} = \vec{E}_1 \cup \cdots \cup \vec{E}_5$, where

$$\vec{E}_1 = \{(x, y) : x \in \mathbb{X}_1, x \neq a_k\},$$

$$\vec{E}_2 = \{(a_k, b_k)\},$$

$$\vec{E}_3 = \{(b_l, a_l) : b_l \in \mathbb{S}'\} = \{(b_k, a_k)\},$$

$$\vec{E}_4 = \{(b_l, a_l) : b_l \in \mathbb{S}''\} = \{(b_j, a_j)\},$$

$$\vec{E}_5 = \{(b_l, a_l) : b_l \in \mathbb{X}_2 \setminus (\mathbb{S}' \cup \mathbb{S}'')\}.$$

Notice that for any $x \in \{a_1, \ldots, a_\nu, b_1, \ldots, b_\nu\} \setminus \{b_j\}$, by the definition of J, x, c_j are in different S-cells, and thus $|\hat{G}_{c_j x}^{(\mathbb{T})}| \leqslant 2M/\sqrt{N\eta}$ by (3.142). Moreover, by the definition of J, for any $y \in \{a_m, b_m : m \in [\![1, \nu]\!] \setminus \{k\}\}$, we have $\text{dist}_{\tilde{\mathcal{G}}^{(\mathbb{T})}}(y, \mathbb{S}') > R/4 - 2r > 2r$ and thus y satisfies the condition either in (3.147) or in (3.148). It follows that $|\tilde{G}_{y c_k}^{(\mathbb{T})}| \leqslant \max\{\Gamma_1, \Gamma_2\} \leqslant \Gamma_1$. For $(x, y) \in \vec{E}_1$. Since $|\vec{E}_1| \leqslant 2\mu \leqslant 4(d-1)^{\ell+1}$, it follows that

$$\sum_{\vec{E}_1} [\cdots] \leqslant 4(d-1)^{\ell+1} \frac{2M}{\sqrt{N\eta}} \frac{\Gamma_1}{\sqrt{d-1}}.$$

For $(x, y) = (a_k, b_k) \in \vec{E}_2$. By (3.144), $\text{dist}_{\tilde{\mathcal{G}}^{(\mathbb{T})}}(b_k, c_k) > 2r$, and thus $|\tilde{G}_{b_k c_k}^{(\mathbb{T})}| \leqslant 2^7 K^3 |m_{sc}| q^r$.

$$\sum_{\vec{E}_2} [\cdots] \leqslant \frac{2M}{\sqrt{N\eta}} \frac{2^7 K^3 |m_{sc}| q^r}{\sqrt{d-1}}.$$

For $(x, y) = (b_k, a_k) \in \vec{E}_3$. By (3.144), $\text{dist}_{\tilde{\mathcal{G}}^{(\mathbb{T})}}(a_k, c_k) > 2r$, and thus $|\tilde{G}_{a_k c_k}^{(\mathbb{T})}| \leqslant 2^7 K^3 |m_{sc}| q^r$.

$$\sum_{\vec{E}_3} [\cdots] \leqslant \frac{2M}{\sqrt{N\eta}} \frac{2^7 K^3 |m_{sc}| q^r}{\sqrt{d-1}}.$$

For $(x, y) = (b_j, a_j) \in \vec{E}_4$, by (3.144) with c_k replaced by c_j, we have the distance estimates

$$\text{dist}_{\tilde{\mathcal{G}}^{(\mathbb{T})}}(c_j, \{a_1, \ldots, a_\mu, b_1, \ldots, b_\nu\}) > 2r.$$

In particular, $\text{dist}_{\hat{\mathcal{G}}^{(\mathbb{T})}}(c_j, b_j) \geqslant \text{dist}_{\tilde{\mathcal{G}}^{(\mathbb{T})}}(c_j, b_j) > 2r$, and $|\hat{G}^{(\mathbb{T})}_{c_j b_j}| \leqslant 2K|m_{sc}|q^r$ by (3.142). Thus

$$\sum_{\vec{E}_4}[\cdots] \leqslant \frac{2K|m_{sc}|q^r}{\sqrt{d-1}}\Gamma_1.$$

For $(x, y) \in \vec{E}_5$, since $|\vec{E}_5| \leqslant \mu \leqslant 2(d-1)^{\ell+1}$, it follows that

$$\sum_{\vec{E}_5}[\cdots] \leqslant 2(d-1)^{\ell+1}\frac{2M}{\sqrt{N\eta}}\frac{\Gamma_1}{\sqrt{d-1}}.$$

The above discussion combined with (3.149) leads to the estimate

$$|\tilde{G}^{(\mathbb{T})}_{c_j c_k}| \leqslant \frac{M}{\sqrt{N\eta}}\left(2 + 2^9 K^3 q^{r+1} + 12(d-1)^{\ell+1/2}\Gamma_1\right) + 2Kq^{r+1}\Gamma_1 \leqslant 2^{12}K^5|m_{sc}|q^{3r+2},$$

provided that $\sqrt{N\eta}q^{3r+2} \geqslant M$. $\qquad\square$

3.8. Stability estimate for the switched graph.

Proposition 3.32. *Under the assumptions of Propositions 3.26, for $\mathbf{S} \in F(\mathcal{G})$ (as in Section 3.6.1) such that $\tilde{\mathcal{G}} = T_{\mathbf{S}}(\mathcal{G}) \in \bar{\Omega}$ (as in Section 3.1.2), the Green's function of the switched graph satisfies the weak stability estimate that for all $i, j \in [\![N]\!]$,*

(3.156)
$$|\tilde{G}_{ij}(z)| \leqslant |\tilde{G}_{jj}(z)| \leqslant 2.$$

Moreover, the off-diagonal entries of the Green's function satisfy the following improved estimates around vertex 1. For all vertices $x \in [\![2, N]\!]$,

(3.157)
$$\left|\tilde{G}_{1x} - P_{1x}(\mathcal{E}_r(1, x, \tilde{\mathcal{G}}))\right| \leqslant (\omega + 1)2^{2\omega+14}K^3|m_{sc}|q^{r+1}.$$

For all estimates, we assume that $\sqrt{d-1} \geqslant \max\{(\omega+1)^2 2^{2\omega+10}, 2^8(\omega+1)K\}$, $\omega'^2 q^\ell \ll 1$ and $\sqrt{N\eta}q^{3r+2} \geqslant M$.

3.8.1. *Preparation of the proof.* As in (3.135), we denote by $\partial_E \mathcal{T}$ the boundary edges of $\mathcal{T}$ in the switched graph $\tilde{\mathcal{G}}$, and the corresponding boundary vertex set by $\mathbb{I} = \{\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_\mu\}$. Let J be the index set of Proposition 3.29. Throughout the following

proof, C represents constants that may differ from line to line, but depends only on the constant K of (3.99) and the excess ω. As in previous proofs, we follow the structure described below (3.71).

Localization. To prove Proposition 3.47, we replace $P_{ij}(\mathcal{E}_r(i,j,\tilde{\mathcal{G}}))$ by a vertex independent Green's function P_{ij} according to Remark 2.8, applied with $\tilde{\mathcal{G}}_0 = \mathcal{B}_{3r}(1,\tilde{\mathcal{G}})$ and $\mathbb{X} = \mathbb{B}_{2r}(1,\tilde{\mathcal{G}})$. We abbreviate

$$\tilde{\mathcal{G}}_1 = \mathrm{TE}(\tilde{\mathcal{G}}_0), \quad \tilde{P} = G(\tilde{\mathcal{G}}_1), \quad \tilde{\mathcal{G}}_1^{(\mathbb{T})} = \mathrm{TE}(\tilde{\mathcal{G}}_0^{(\mathbb{T})}), \quad \tilde{P}^{(\mathbb{T})} = G(\tilde{\mathcal{G}}_1^{(\mathbb{T})}),$$

Notice that $\tilde{\mathcal{G}}_1^{(\mathbb{T})}$ is the same as removing the vertices $\mathbb{T}$ from $\tilde{\mathcal{G}}_1$, and thus $\tilde{P}^{(\mathbb{T})} = G^{(\mathbb{T})}(\tilde{\mathcal{G}}_1)$.

Claim 3.33. *Let $k \in J$ (as in Section 3.7.1), and let $\mathbb{K}$ be the connected component of $\tilde{\mathcal{G}}_0$ containing $\tilde{a}_k = c_k$. Then*

$$(3.158) \qquad\qquad \{m \in [\![1,\mu]\!] : \tilde{a}_m \in \mathbb{K}\} = \{k\},$$

and

$$(3.159) \qquad\qquad \max_{i \in \mathbb{K}} \mathrm{dist}_{\tilde{\mathcal{G}}^{(\mathbb{T})}}(i, \tilde{a}_k) \leqslant 3r.$$

Proof. (3.158) follows from the condition (3.139) in the definition of J, i.e. from $\mathrm{dist}_{\tilde{\mathcal{G}}^{(\mathbb{T})}}(\tilde{a}_k, \{\tilde{a}_m : m \in [\![1,\mu]\!] \setminus \{k\}\}) > R/4 > 6r$. (3.159) follows from (3.158) and the construction of the subgraph $\tilde{\mathcal{G}}_0$. $\qquad\square$

Verification of assumptions in Proposition 2.7. As subgraphs of $\tilde{\mathcal{G}}$, both $\tilde{\mathcal{G}}_0$ and $\tilde{\mathcal{G}}_0^{(\mathbb{T})}$ have excess at most ω. The deficit function g of $\tilde{\mathcal{G}}_0$ vanishes. By Proposition 3.10, on each connected components of $\tilde{\mathcal{G}}_0^{(\mathbb{T})}$, the deficit function obeys $\sum g(v) \leqslant \omega + 1 \leqslant 8\omega$. Thus the assumptions for (2.14) are verified for both graphs $\tilde{\mathcal{G}}_0$ and $\tilde{\mathcal{G}}_0^{(\mathbb{T})}$, and we have

(2.16):

$$(3.160) \qquad \left| P_{ij}(\mathcal{E}_r(i,j,\tilde{\mathcal{G}})) - \tilde{P}_{ij} \right|, \left| P_{ij}(\mathcal{E}_r(i,j,\tilde{\mathcal{G}}^{(\mathbb{T})})) - \tilde{P}_{ij}^{(\mathbb{T})} \right| \leqslant 2^{2\omega+3} |m_{sc}| q^{r+1}$$

for $i, j \in \mathbb{X}$, provided that $\sqrt{d-1} \geqslant 2^{\omega+2}$.

Starting point. The normalized adjacency matrices of $\tilde{\mathcal{G}}$ and $\tilde{\mathcal{G}}_1$ respectively have the block form

$$\begin{bmatrix} H & \tilde{B}' \\ \tilde{B} & D \end{bmatrix}, \quad \begin{bmatrix} H & \tilde{B}'_1 \\ \tilde{B}_1 & D_1 \end{bmatrix},$$

where H is the normalized adjacency matrix for $\mathcal{T}$, and $\tilde{B}$ (respectively $\tilde{B}_1$) corresponds to the edges from $\mathbb{I}$ to $\mathbb{T}_\ell$, where $\mathbb{I}$ is the set of boundary vertices of $\mathcal{T}$ in the switched graph $\tilde{\mathcal{G}}$ as defined in (3.135), and $\mathbb{T}_\ell$ is the inner vertex boundary of $\mathcal{T}$ as in (3.47). To be precise, the nonzero entries of $\tilde{B}$ and $\tilde{B}_1$ occur for the indices $(i,j) \in \mathbb{I} \times \mathbb{T}_\ell$ and take values $1/\sqrt{d-1}$. Notice that $\tilde{B}_{ij} = (\tilde{B}_1)_{ij}$; in the rest of this section we will therefore not distinguish B and $\tilde{B}$.

By the Schur Complement formula (B.3), we have

$$(3.161) \qquad \tilde{G}|_{\mathbb{T}} = (H - z - \tilde{B}' \tilde{G}^{(\mathbb{T})} \tilde{B})^{-1},$$

$$(3.162) \qquad \tilde{P}|_{\mathbb{T}} = (H - z - \tilde{B}' \tilde{P}^{(\mathbb{T})} \tilde{B})^{-1},$$

and, by the resolvent identity (B.1), the difference of (3.161) and (3.162) is

$$(3.163) \qquad \tilde{G}|_{\mathbb{T}} - \tilde{P}|_{\mathbb{T}} = (\tilde{G} - \tilde{P}) \tilde{B}' (\tilde{G}^{(\mathbb{T})} - \tilde{P}^{(\mathbb{T})}) \tilde{B} \tilde{P} + \tilde{P} \tilde{B}' (\tilde{G}^{(\mathbb{T})} - \tilde{P}^{(\mathbb{T})}) \tilde{B} \tilde{P}.$$

In terms of the random walk heuristic described in Section **??**, (3.163) has the interpretation that only walks that exit $\mathbb{T}$ contribute (see Figure 11). We will adopt suggestive terminology corresponding to the random walk picture below. By Proposition 3.29, the Green's function $\tilde{G}^{(\mathbb{T})}$ is small between most vertices in $\mathbb{I}$. This is the
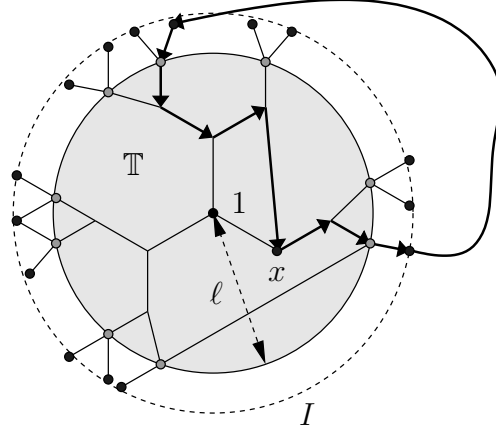
FIGURE 11. Only walks that exit $\mathbb{T}$ contribute to (3.163).

main reason that the right-hand side of (3.163) is small. In the following, we analyze the various contributions precisely.

3.8.2. *Boundary.* The following lemma estimates the weight of "walks" from $x \in \mathbb{T}$ to $\mathbb{T}_\ell$, the inner vertex boundary of $\mathcal{T}$. It depends on the distance of x to the boundary, or equivalently that from x to 1.

Lemma 3.34. *Assume that $\tilde{\mathcal{G}}_0$ has excess at most ω. For vertices $x \in \mathbb{T}_{\ell_1}$, i.e. x is at distance ℓ_1 from vertex 1, we have*

$$(3.164) \qquad \sum_{k \in [\![1,\mu]\!]} |\tilde{P}_{l_k x}| \leqslant (\omega+1)2^{\omega+3}(\ell_1+1)|m_{sc}|^{\ell-\ell_1+1}(d-1)^{(\ell-\ell_1)/2+1}.$$

For vertices $x \in \mathbb{T}_{\ell_1}$ and $y \in \mathbb{T}_{\ell_2}$, with $\ell_1 \geqslant \ell_2$, we have

$$(3.165) \qquad \sum_{k \in [\![1,\mu]\!]} |\tilde{P}_{l_k x}||\tilde{P}_{l_k y}| \leqslant \frac{(\omega+1)^2 2^{2\omega+6}(\ell_1-\ell_2+2)}{(d-1)^{(\ell_1-\ell_2)/2-1}}|m_{sc}|^{2\ell-\ell_1-\ell_2+2}.$$

The proof of the lemma uses the following combinatorial estimate on the distances of a vertex x to $\mathbb{T}_\ell$ (which is the inner vertex boundary of $\mathcal{T}$).

Lemma 3.35. *Assume that the graph $\tilde{\mathcal{G}}_0 = \mathcal{B}_{3r}(1, \mathcal{G})$ has excess at most ω. Given $x \in \mathbb{T}_{\ell_1}$, let L_x be the multiset consisting of $2(\omega+1)(d-1)^{\ell-\ell_3}$ copies of the number*

$q^{\ell+\ell_1-2\ell_3}$ *for $\ell_3 \in [\![0,\ell_1]\!]$, and let K_x be the multiset $K_x = \{q^{\mathrm{dist}_{\tilde{\mathcal{G}}_0}(x,i)} : i \in \mathbb{T}_\ell\}$. Then the k-th largest number of K_x is smaller than or equal to the k-th largest number of L_x.*

We postpone the proof of the lemma to Appendix A.3. Given the lemma, the proof of Lemma 3.34 is completed as follows.

Proof of Lemma 3.34. To prove (3.164), we use

$$\sum_{k\in[\![1,\mu]\!]} |\tilde{P}_{l_k x}| \leqslant (d-1)\sum_{i\in\mathbb{T}_\ell} |\tilde{P}_{ix}| \leqslant 2^{\omega+2}(d-1)|m_{sc}|\sum_{i\in\mathbb{T}_\ell} q^{\mathrm{dist}_{\tilde{\mathcal{G}}_0}(x,i)},$$

by Proposition 2.7. Defining the multiset L_x as in Lemma 3.35, the inequality continuous with

$$2^{\omega+2}(d-1)|m_{sc}|\sum_{i\in\mathbb{T}_\ell} q^{\mathrm{dist}_{\tilde{\mathcal{G}}_0}(x,i)} \leqslant 2^{\omega+2}(d-1)|m_{sc}|(2\omega+2)\sum_{\ell_3=0}^{\ell_1}(d-1)^{\ell-\ell_3}q^{\ell+\ell_1-2\ell_3}$$

$$\leqslant (\omega+1)2^{\omega+3}(\ell_1+1)|m_{sc}|^{\ell-\ell_1+1}(d-1)^{(\ell-\ell_1)/2+1}.$$

This finishes the proof of (3.164).

For the proof of (3.165), we use

$$\sum_{k\in[\![1,\mu]\!]} |\tilde{P}_{l_k x}||\tilde{P}_{l_k y}| \leqslant (d-1)\sum_{i\in\mathbb{T}_\ell} |\tilde{P}_{ix}||\tilde{P}_{iy}| \leqslant 2^{2\omega+4}|m_{sc}|^2(d-1)\sum_{i\in\mathbb{T}_\ell} q^{\mathrm{dist}_{\tilde{\mathcal{G}}_0}(x,i)}q^{\mathrm{dist}_{\tilde{\mathcal{G}}_0}(y,i)}.$$

We define the multisets L_x and L_y as in Lemma 3.35. More precisely, L_x consists of $2(\omega+1)(d-1)^{\ell-\ell_3}$ copies of $q^{\ell+\ell_1-2\ell_3}$ for $\ell_3 \in [\![0,\ell_1]\!]$, and L_y consists of $2(\omega+1)(d-$

$1)^{\ell-\ell_3}$ copies of $q^{\ell+\ell_2-2\ell_3}$ for $\ell_3 \in [\![0, \ell_2]\!]$. By the rearrangement inequality, we have

$$\sum_{i \in \mathbb{T}_\ell} q^{\text{dist}_{\tilde{\mathcal{G}}_0}(x,i)} q^{\text{dist}_{\tilde{\mathcal{G}}_0}(y,i)}$$

$$\leqslant 4(\omega+1)^2 \left(\sum_{\ell_3=0}^{\ell_2} (d-1)^{\ell-\ell_3} q^{\ell+\ell_1-2\ell_3} q^{\ell+\ell_2-2\ell_3} + \sum_{\ell_3=\ell_2+1}^{\ell_1} (d-1)^{\ell-\ell_3} q^{\ell+\ell_1-2\ell_3} q^{\ell-\ell_2} \right)$$

$$\leqslant \frac{4(\omega+1)^2(\ell_1 - \ell_2 + 2)}{(d-1)^{(\ell_1-\ell_2)/2}} |m_{sc}|^{2\ell-\ell_1-\ell_2}.$$

This finishes the proof of (3.165). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 3.36. *In the worst case, when $x = 1$, we have*

$$(3.166) \qquad\qquad \sum_{k \in [\![1,\mu]\!]} |\tilde{P}_{l_k x}| \leqslant (\omega+1)2^{\omega+3}|m_{sc}|^{\ell+1}(d-1)^{\ell/2+1}.$$

Moreover, when $x, y \in \mathbb{T}_{\ell_1}$, we have

$$(3.167) \qquad\qquad \sum_{k \in [\![1,\mu]\!]} |\tilde{P}_{l_k x}||\tilde{P}_{l_k y}| \leqslant (\omega+1)^2 2^{2\omega+7}|m_{sc}|^{2\ell-2\ell_1+2}(d-1).$$

These special cases will be used below.

3.8.3. *Outside $\mathbb{T}$.* The following proposition shows that the weight of "walks" outside $\mathbb{T}$ is small. It essentially follows from Proposition 3.29.

Proposition 3.37. *Under the assumptions of Proposition 3.32, for any vertex $j \in [\![N]\!] \setminus \mathbb{T}$ such that $\text{dist}_{\tilde{\mathcal{G}}}(1, j) \leqslant 2r$, we have*

$$(3.168) \qquad\qquad \sum_{k \in [\![1,\mu]\!]} |\tilde{G}_{\tilde{a}_k j}^{(\mathbb{T})} - \tilde{P}_{\tilde{a}_k j}^{(\mathbb{T})}| \leqslant C\omega'|m_{sc}|q^r.$$

Moreover, for any vertex $j \in [\![N]\!] \setminus \mathbb{T}$ such that $\text{dist}_{\tilde{\mathcal{G}}}(1, j) > 2r$, we have

$$(3.169) \qquad\qquad \sum_{k \in [\![1,\mu]\!]} |\tilde{G}_{\tilde{a}_k j}^{(\mathbb{T})}| \leqslant C\omega'|m_{sc}|q^r,$$

and

(3.170)
$$\sum_{k \neq m \in \llbracket 1, \mu \rrbracket} |\tilde{G}_{\tilde{a}_k \tilde{a}_m}^{(\mathbb{T})} - \tilde{P}_{\tilde{a}_k \tilde{a}_m}^{(\mathbb{T})}| \leqslant C\omega'^2 |m_{sc}| q^r,$$

where the constants C depend only on the excess ω and K (from Proposition 3.26). For all estimates, we assume $\sqrt{d-1} \geqslant \max\{(\omega+1)^2 2^{2\omega+10}, 2^8(\omega+1)K\}$, $\omega'q^r \ll 1$ and $\sqrt{N\eta}q^{3r+2} \geqslant M$.

Claim 3.38. *Let $j \in \llbracket N \rrbracket \setminus \mathbb{T}$ be as in the statement of Proposition 3.37. Then*

(3.171)
$$|\{k \in \llbracket 1, \nu \rrbracket : \text{dist}_{\tilde{\mathcal{G}}^{(\mathbb{T})}}(j, a_k) \leqslant R/4\}| \leqslant 5\omega,$$

(3.172)
$$|\{k \in \llbracket 1, \nu \rrbracket : \text{dist}_{\tilde{\mathcal{G}}^{(\mathbb{T})}}(j, \tilde{a}_k) \leqslant R/2\}| \leqslant \omega + 1.$$

Proof. The first claim follows from (3.54). The second one follows from (3.30) by considering the graph $\tilde{\mathcal{G}}$, since by our assumption $\tilde{\mathcal{G}} \in \bar{\Omega}$, the R-neighborhood $\mathcal{B}_R(1, \tilde{\mathcal{G}})$ has excess at most ω. $\square$

Proof of Proposition 3.37. Recall the index set $J \subset \llbracket 1, \nu \rrbracket$ defined previously in Section 3.7.1. To prove (3.168), we decompose $\llbracket 1, \mu \rrbracket$ according to the relations between $\{a_k, b_k, c_k\}$ and vertex j as $\llbracket 1, \mu \rrbracket = J_1 \cup J_2$, where

$$J_1 = \{k \in J : j \nsim b_k, \text{ dist}_{\tilde{\mathcal{G}}^{(\mathbb{T})}}(j, a_k) \geqslant 2r, \text{ and } \text{dist}_{\tilde{\mathcal{G}}^{(\mathbb{T})}}(j, \tilde{a}_k) \geqslant R/2\},$$

$$J_2 = \llbracket 1, \mu \rrbracket \setminus J_1.$$

By the defining relation (3.103) of $F(\mathcal{G})$ and Proposition 3.29, we have $|J| \geqslant \nu - \omega' - 6\omega \geqslant \mu - \omega' - 9\omega$. Combining with (3.171), (3.172) and (3.61), which states that $|\{k \in \llbracket 1, \nu \rrbracket : j \sim b_k\}| < \omega'$, we get

$$|J_1| \geqslant \mu - 2\omega' - 15\omega, \quad |J_2| \leqslant 2\omega' + 15\omega.$$

Bounding by the total number of terms, we also have $|J_1| \leqslant \mu \leqslant 2(d-1)^{\ell+1}$. Now, for $k \in J_1$, we have $\tilde{a}_k = c_k$ and the conditions for (3.138) are satisfied. Moreover, for $k \in J_1$, we have $\mathrm{dist}_{\tilde{\mathcal{G}}^{(\mathbb{T})}}(\tilde{a}_k, j) \geqslant R/2$, and therefore by (3.159) the vertices $\tilde{a}_k$ and j are in different connected components of $\tilde{\mathcal{G}}_0^{(\mathbb{T})}$; it follows that $|\tilde{P}_{\tilde{a}_k j}^{(\mathbb{T})}| = 0$. Therefore, by (3.138),

$$(3.173) \qquad \sum_{k \in J_1} |\tilde{G}_{\tilde{a}_k j}^{(\mathbb{T})} - \tilde{P}_{\tilde{a}_k j}^{(\mathbb{T})}| = \sum_{k \in J_1} |\tilde{G}_{\tilde{a}_k j}^{(\mathbb{T})}| \leqslant 2(d-1)^{\ell+1}(2^{12} K^5 |m_{sc}| q^{2r+1}) \leqslant 2^{13} K^5 |m_{sc}| q^r.$$

For $k \in J_2$, by (3.102) and (3.160),

$$(3.174) \qquad \sum_{k \in J_2} |\tilde{G}_{\tilde{a}_k j}^{(\mathbb{T})} - \tilde{P}_{\tilde{a}_k j}^{(\mathbb{T})}| \leqslant (2\omega' + 16\omega)(2^7 K^3 + 2^{2\omega+3} q)|m_{sc}| q^r.$$

Then (3.168) follows by combining (3.173) and (3.174).

For (3.169), again, we split the sum over J_1 and over J_2 as above. For $k \in J_1$, similarly to (3.168), we have

$$(3.175) \qquad \sum_{k \in J_1} |\tilde{G}_{\tilde{a}_k j}^{(\mathbb{T})}| \leqslant 2(d-1)^{\ell+1} 2^{12} K^5 |m_{sc}| q^{2r+1} \leqslant 2^{13} K^5 |m_{sc}| q^r.$$

For $k \in J_2$, we note that $\mathrm{dist}_{\tilde{\mathcal{G}}}(\tilde{a}_k, j) \geqslant \mathrm{dist}_{\tilde{\mathcal{G}}}(1, j) - \mathrm{dist}_{\tilde{\mathcal{G}}}(1, \tilde{a}_k) \geqslant 2r - \ell > r$, so that $P_{\tilde{a}_k j}(\mathcal{E}_r(\tilde{a}_k, j, \tilde{\mathcal{G}}^{(\mathbb{T})})) = 0$. Therefore, by (3.102), we have

$$(3.176) \qquad \sum_{k \in J_2} |\tilde{G}_{\tilde{a}_k j}^{(\mathbb{T})}| \leqslant (2\omega' + 15\omega) 2^7 K^3 |m_{sc}| q^r.$$

Again (3.169) follows by combining (3.175) and (3.176).

For (3.170), we split the sum over

$$\{k \neq m \in [\![1, \mu]\!]\} = \{k \neq m \in [\![1, \mu]\!] \setminus J\} \cup \{k \in [\![1, \mu]\!] \setminus J, m \in J\}$$

$$\cup \{k \in J, m \in [\![1, \mu]\!] \setminus J\} \cup \{k \neq m \in J\}.$$

Since $|[\![1, \mu]\!] \setminus J| \leqslant \omega' + 9\omega$, for $k \neq m \in [\![1, \mu]\!] \setminus J$, by Proposition 3.26 and (3.160),

$$\sum_{k \neq m \in [\![1, \mu]\!] \setminus J} |\tilde{G}^{(\mathbb{T})}_{\tilde{a}_k \tilde{a}_m} - \tilde{P}^{(\mathbb{T})}_{\tilde{a}_k \tilde{a}_m}| \leqslant (\omega' + 9\omega)^2 (2^7 K^3 + 2^{2\omega+3} q) |m_{sc}| q^r.$$

For $k \in [\![1, \mu]\!] \setminus J, m \in J$, by (3.158) $\tilde{a}_k$ and $\tilde{a}_m$ are in different connected components of $\tilde{\mathcal{G}}^{(\mathbb{T})}_0$, and thus $|\tilde{P}^{(\mathbb{T})}_{\tilde{a}_k \tilde{a}_m}| = 0$. By (3.136),

$$\sum_{k \in [\![1, \mu]\!] \setminus J, m \in J} |\tilde{G}^{(\mathbb{T})}_{\tilde{a}_k \tilde{a}_m} - \tilde{P}^{(\mathbb{T})}_{\tilde{a}_k \tilde{a}_m}| = \sum_{k \in [\![1, \mu]\!] \setminus J, m \in J} |\tilde{G}^{(\mathbb{T})}_{ij}| \leqslant (\omega' + 9\omega) 2(d-1)^{\ell+1} (2^9 K^4 |m_{sc}| q^{2r+1})$$

$$\leqslant (\omega' + 9\omega) 2^{10} K^4 |m_{sc}| q^r.$$

The same estimate holds for $k \in J, m \in [\![1, \mu]\!] \setminus J$. For $k \neq m \in J$, the same reasoning as above gives $\tilde{P}^{(\mathbb{T})}_{\tilde{a}_k \tilde{a}_m} = 0$. By (3.137) and noticing that $|J| \leqslant \mu \leqslant 2(d-1)^{\ell+1}$, we have

$$\sum_{k \neq m \in J} |\tilde{G}^{(\mathbb{T})}_{\tilde{a}_k \tilde{a}_m} - \tilde{P}^{(\mathbb{T})}_{\tilde{a}_k \tilde{a}_m}| = \sum_{k \neq m \in J} |\tilde{G}^{(\mathbb{T})}_{\tilde{a}_k \tilde{a}_m}| \leqslant 4(d-1)^{2\ell+2} 2^{12} K^5 |m_{sc}| q^{3r+2} \leqslant 2^{14} K^5 |m_{sc}| q^r.$$

Now (3.170) follows by combining the above four cases. $\qquad\square$

3.8.4. *Proof of* (3.157). The proof of (3.157) follows essentially from (3.163) and the fact the difference of $\tilde{G}^{(\mathbb{T})}$ and $\tilde{P}^{(\mathbb{T})}$ is small (Proposition 3.26).

Claim 3.39. *For all $x \in \mathbb{T}$,*

$$|\tilde{G}_{1x} - \tilde{P}_{1x}| \leqslant (\omega + 1) 2^{2\omega+13} K^3 |m_{sc}| q^r.$$

Moreover, for $x \in \mathbb{T} \setminus \{1\}$, we have the stronger estimate

(3.177) $$|\tilde{G}_{1x} - \tilde{P}_{1x}| \leqslant (\omega + 1) 2^{2\omega+14} K^3 |m_{sc}| q^{r+1}.$$

Proof. Let $\Gamma_1 = \max_{x \in \mathbb{T}} |\tilde{G}_{1x} - \tilde{P}_{1x}|$. Then the first term on the right-hand side of (3.163) is bounded by

(3.178)

$$
|((\tilde{G} - \tilde{P})\tilde{B}'(\tilde{G}^{(\mathbb{T})} - \tilde{P}^{(\mathbb{T})})\tilde{B}\tilde{P})_{1x}| \leqslant \frac{\Gamma_1}{d-1} \sum_{k \in [\![1,\mu]\!]} \sum_{m \in [\![1,\mu]\!]} |\tilde{G}^{(\mathbb{T})}_{\tilde{a}_k \tilde{a}_m} - \tilde{P}^{(\mathbb{T})}_{\tilde{a}_k \tilde{a}_m}||\tilde{P}_{l_m x}|
$$

$$
\leqslant \frac{\Gamma_1(C\omega' q^{r+1})}{\sqrt{d-1}} \sum_{m \in [\![1,\mu]\!]} |\tilde{P}_{l_m x}| \leqslant \Gamma_1(C\omega' q^{r+1})((\omega+1)2^{\omega+3}|m_{sc}|^{\ell+1}(d-1)^{(\ell+1)/2})
$$

$$
\leqslant C\omega'|m_{sc}|q^{\ell+1}\Gamma_1,
$$

where we used (3.168) and (3.166). Next we bound the second term on the right-hand side of (3.163). For, say $x \in \mathbb{T}_{\ell_1}$, we have

(3.179)

$$
|(\tilde{P}\tilde{B}'(\tilde{G}^{(\mathbb{T})} - \tilde{P}^{(\mathbb{T})})\tilde{B}\tilde{P})_{1x}| \leqslant \frac{1}{d-1} \sum_{k \in [\![1,\mu]\!]} |\tilde{P}_{1l_k}||\tilde{G}^{(\mathbb{T})}_{\tilde{a}_k \tilde{a}_k} - \tilde{P}^{(\mathbb{T})}_{\tilde{a}_k \tilde{a}_k}||\tilde{P}_{l_k x}|
$$

$$
+ \frac{1}{d-1} \sum_{k \neq m \in [\![1,\mu]\!]} |\tilde{P}_{1l_k}||\tilde{G}^{(\mathbb{T})}_{\tilde{a}_k \tilde{a}_m} - \tilde{P}^{(\mathbb{T})}_{\tilde{a}_k \tilde{a}_m}||\tilde{P}_{l_m x}|.
$$

By (2.12) and (2.15), for any $k, m \in [\![1, \mu]\!]$,

(3.180) $$|\tilde{P}_{1l_k}| \leqslant 2^{\omega+2}|m_{sc}|q^\ell, \quad |\tilde{P}_{l_m x}| \leqslant 2|m_{sc}|.$$

We can estimate the first term in (3.179) in the following way:

$$
\frac{1}{d-1} \sum_{k \in [\![1,\mu]\!]} |\tilde{P}_{1l_k}||\tilde{G}^{(\mathbb{T})}_{\tilde{a}_k \tilde{a}_k} - \tilde{P}^{(\mathbb{T})}_{\tilde{a}_k \tilde{a}_k}||\tilde{P}_{l_k x}| \leqslant 2^{\omega+2}q^{\ell+1}(2^7 K^3 + 2^{2\omega+3}q)q^{r+1} \sum_{k \in [\![1,\mu]\!]} |\tilde{P}_{l_k x}|
$$

$$
\leqslant (\omega+1)2^{2\omega+5}(\ell_1+1)(2^7 K^3 + 2^{2\omega+3}q)q^{r+\ell_1},
$$

where in the first inequality we used (3.180), (3.102) and (3.160), in the second inequality we used estimate (3.164). For the second term in (3.179), we have

$$\frac{1}{d-1} \sum_{k \neq m \in [\![1,\mu]\!]} |\tilde{P}_{1l_k}| |\tilde{G}^{(\mathbb{T})}_{\tilde{a}_k \tilde{a}_m} - \tilde{P}^{(\mathbb{T})}_{\tilde{a}_k \tilde{a}_m}| |\tilde{P}_{l_m x}|$$

$$\leqslant 2^{\omega+2} q^{\ell+1} \sum_{k \neq m \in [\![1,\mu]\!]} |\tilde{G}^{(\mathbb{T})}_{\tilde{a}_k \tilde{a}_m} - \tilde{P}^{(\mathbb{T})}_{\tilde{a}_k \tilde{a}_m}| (2q) \leqslant C \omega'^2 q^{r+\ell+2},$$

where we used (3.170) and (3.180). It follows that

(3.181)
$$|\tilde{G}_{1x} - \tilde{P}_{1x}| \leqslant C \omega' |m_{sc}| q^{\ell+1} \Gamma_1 + (\omega + 1) 2^{2\omega+5} (\ell_1 + 1)(2^7 K^3 + 2^{2\omega+3} q) q^{r+\ell_1} + C \omega'^2 q^{r+\ell+2}.$$

By taking the maximum over $x \in \mathbb{T}$ and rearranging it, we have $\Gamma_1 \leqslant (\omega+1) 2^{2\omega+13} K^3 |m_{sc}| q^r$, provided that $\omega'^2 q^\ell \ll 1$ and $\sqrt{d-1} \geqslant (\omega + 1)^2 2^{2\omega+10}$.

For (3.177), it follows from (3.181), the estimate of Γ_1 and $\ell_1 \geqslant 1$,

$$|\tilde{G}_{1x} - \tilde{P}_{1x}| \leqslant C \omega' q^{r+\ell+1} + (\omega + 1) 2^{2\omega+6} (2^7 K^3 + 2^{2\omega+3} q) q^{r+1} + C \omega'^2 q^{r+\ell+2}$$

(3.182)
$$\leqslant (\omega + 1) 2^{2\omega+6} (2^7 K^3 + 1) q^{r+1},$$

provided that $\omega'^2 q^\ell \ll 1$ and $\sqrt{d-1} \geqslant (\omega + 1)^2 2^{2\omega+10}$. $\qquad \square$

Proof of (3.157). For $x \in \mathbb{T} \setminus \{1\}$, the estimate (3.157) follows from (3.182) and (3.160):

$$\left| \tilde{G}_{1x} - P_{1x}(\mathcal{E}_r(1, x, \tilde{\mathcal{G}})) \right| \leqslant \left| \tilde{G}_{1x} - \tilde{P}_{1x} \right| + \left| P_{1x}(\mathcal{E}_r(1, x, \tilde{\mathcal{G}})) - \tilde{P}_{1x} \right|$$

$$\leqslant (\omega + 1) 2^{2\omega+6} (2^7 K^3 + 1) q^{r+1} + 2^{2\omega+3} |m_{sc}| q^{r+1} \leqslant (\omega + 1) 2^{2\omega+14} K^3 |m_{sc}| q^{r+1}.$$

Thus it only remains to prove (3.157) for $x \notin \mathbb{T}$.

For $x \in \mathbb{B}_{2r}(1, \tilde{\mathcal{G}}) \setminus \mathbb{T}$, we have by the Schur complement formula (B.4):

$$\tilde{G} = -\tilde{G}\tilde{B}'\tilde{G}^{(\mathbb{T})}, \quad \tilde{P} = -\tilde{P}\tilde{B}'\tilde{P}^{(\mathbb{T})}.$$

Therefore, by taking the difference of these two equations,

(3.183)
$$|\tilde{G}_{1x} - \tilde{P}_{1x}| \leqslant \frac{1}{\sqrt{d-1}} \sum_{k \in [\![1,\mu]\!]} |\tilde{G}_{1l_k}||\tilde{P}_{\tilde{a}_k x}^{(\mathbb{T})} - \tilde{G}_{\tilde{a}_k x}^{(\mathbb{T})}| + \frac{1}{\sqrt{d-1}} \sum_{k \in [\![1,\mu]\!]} |\tilde{G}_{1l_k} - \tilde{P}_{1l_k}||\tilde{P}_{\tilde{a}_k x}^{(\mathbb{T})}|.$$

For the first term in (3.183), notice that by combining (3.180) and (3.177), we have

(3.184)
$$|\tilde{G}_{1l_k}| \leqslant |\tilde{G}_{1l_k} - \tilde{P}_{1l_k}| + |\tilde{P}_{1l_k}| \leqslant 2^{\omega+3}|m_{sc}|q^\ell.$$

The first term in (3.183) is bounded by

$$\frac{1}{\sqrt{d-1}} \sum_{k \in [\![1,\mu]\!]} |\tilde{G}_{1l_k}||\tilde{P}_{\tilde{a}_k x}^{(\mathbb{T})} - \tilde{G}_{\tilde{a}_k x}^{(\mathbb{T})}| \leqslant Cq^{\ell+1} \sum_{k \in [\![1,\mu]\!]} |\tilde{P}_{\tilde{a}_k x}^{(\mathbb{T})} - \tilde{G}_{\tilde{a}_k x}^{(\mathbb{T})}| \leqslant C\omega' q^{r+\ell+1},$$

where we used (3.168). For the second term in (3.183), since $\tilde{\mathcal{G}} \in \bar{\Omega}$, its radius-$R$ neighborhood of vertex 1 has excess at most ω. By (3.30) there are at most $\omega + 1$ indices $k \in [\![1, \mu]\!]$, such that $\tilde{a}_k$ is in the same connected component as x in the graph $\tilde{\mathcal{G}}_0$. Thus, $\tilde{P}_{\tilde{a}_k x}^{(\mathbb{T})}$ are zero for all $k \in [\![1, \mu]\!]$ except for at most $\omega + 1$ of them, and they are bounded $|\tilde{P}_{\tilde{a}_k x}^{(\mathbb{T})}| \leqslant 2|m_{sc}|$ by (2.15).

$$\frac{1}{\sqrt{d-1}} \sum_{k \in [\![1,\mu]\!]} |\tilde{G}_{1l_k} - \tilde{P}_{1l_k}||\tilde{P}_{\tilde{a}_k x}^{(\mathbb{T})}| \leqslant (\omega+1)^2 2^{2\omega+15} K^3 |m_{sc}|q^{r+2},$$

where we used (3.177). Combining the arguments above, they lead to

$$|\tilde{G}_{1x} - \tilde{P}_{1x}| \leqslant 2^5 K^3 |m_{sc}|q^{r+1} + C\omega' q^{r+\ell+1},$$

given that $\sqrt{d-1} \geqslant (\omega+1)^2 2^{2\omega+10}$. The estimate (3.157) for $x \in \mathbb{B}_{2r}(1, \tilde{\mathcal{G}}) \backslash \mathbb{T}$ follows, provided $\omega'^2 q^\ell \ll 1$.

For $x \notin \mathbb{B}_{2r}(1, \tilde{\mathcal{G}})$, we have

(3.185)
$$|\tilde{G}_{1x}| \leqslant \frac{1}{\sqrt{d-1}} \sum_{k \in [\![1,\mu]\!]} |\tilde{G}_{1l_k}||\tilde{G}_{\tilde{a}_k x}^{(\mathbb{T})}| \leqslant 2^{\omega+3}q^{\ell+1} \sum_{k \in [\![1,\mu]\!]} |\tilde{G}_{\tilde{a}_k x}^{(\mathbb{T})}| \leqslant C\omega' q^{r+\ell+1} \ll q^{r+1},$$

where we used (3.184) and (3.169). This finishes the proof of (3.157). $\qquad\square$

3.8.5. *Proof of* (3.156).

Proof of (3.156). For $x, y \in \mathbb{T}$, we denote $\Gamma = \max_{x,y\in\mathbb{T}}\{|\tilde{G}_{xy} - \tilde{P}_{xy}|\}$. Then, by the Schur complement formula (3.163),

(3.186)
$$|\tilde{G}_{xy} - \tilde{P}_{xy}| \leqslant |((\tilde{G} - \tilde{P})\tilde{B}'(\tilde{G}^{(\mathbb{T})} - \tilde{P}^{(\mathbb{T})})\tilde{B}\tilde{P})_{xy}| + |(\tilde{P}\tilde{B}'(\tilde{G}^{(\mathbb{T})} - \tilde{P}^{(\mathbb{T})})\tilde{B}\tilde{P})_{xy}|.$$

The estimate of the first term follows the same argument as that for (3.178):

$$|((\tilde{G} - \tilde{P})\tilde{B}'(\tilde{G}^{(\mathbb{T})} - \tilde{P}^{(\mathbb{T})})\tilde{B}\tilde{P})_{xy}| \leqslant C\omega'|m_{sc}|q^{\ell+1}\Gamma \leqslant \Gamma/2.$$

For the second term, similarly, we have

$$\left|(\tilde{P}\tilde{B}'(\tilde{G}^{(\mathbb{T})} - \tilde{P}^{(\mathbb{T})})\tilde{B}\tilde{P})_{xy}\right|$$
$$\leqslant \frac{1}{d-1}\sum_{k\in[\![1,\mu]\!]}|\tilde{P}_{xl_k}||\tilde{G}^{(\mathbb{T})}_{\tilde{a}_k\tilde{a}_k} - \tilde{P}^{(\mathbb{T})}_{\tilde{a}_k\tilde{a}_k}||\tilde{P}_{l_ky}| + \frac{1}{d-1}\sum_{k\neq m\in[\![1,\mu]\!]}|\tilde{P}_{xl_k}||\tilde{G}^{(\mathbb{T})}_{\tilde{a}_k\tilde{a}_m} - \tilde{P}^{(\mathbb{T})}_{\tilde{a}_k\tilde{a}_m}||\tilde{P}_{l_my}|$$
$$\leqslant \frac{C|m_{sc}|q^r}{d-1}\sum_{k\in[\![1,\mu]\!]}|\tilde{P}_{xl_k}||\tilde{P}_{l_ky}| + (Cq)^2\sum_{k\neq m\in[\![1,\mu]\!]}|\tilde{G}^{(\mathbb{T})}_{\tilde{a}_k\tilde{a}_m} - \tilde{P}^{(\mathbb{T})}_{\tilde{a}_k\tilde{a}_m}| \leqslant C\omega'^2|m_{sc}|q^r,$$

where we bounded $|\tilde{P}_{xl_k}|, |\tilde{P}_{l_my}| \leqslant C|m_{sc}|$ and used the estimates (3.102), (3.167) and (3.170). Therefore, by taking supremum of both sides of (3.186) and rearranging, we have $\Gamma \leqslant C\omega'^2|m_{sc}|q^r$.

For $x \in \mathbb{T}$ and $y \in \mathbb{B}_{2r}(1, \tilde{\mathcal{G}}) \setminus \mathbb{T}$, the same argument as for (3.183) implies:

$$|\tilde{G}_{xy} - \tilde{P}_{xy}| \leqslant \frac{1}{\sqrt{d-1}}\sum_{k\in[\![1,\mu]\!]}|\tilde{G}_{xl_k}||\tilde{P}^{(\mathbb{T})}_{\tilde{a}_kx} - \tilde{G}^{(\mathbb{T})}_{\tilde{a}_kx}| + \frac{1}{\sqrt{d-1}}\sum_{k\in[\![1,\mu]\!]}|\tilde{G}_{xl_k} - \tilde{P}_{xl_k}||\tilde{P}^{(\mathbb{T})}_{\tilde{a}_kx}|.$$
$$\leqslant Cq\sum_{k\in[\![1,\mu]\!]}|\tilde{P}^{(\mathbb{T})}_{\tilde{a}_ky} - \tilde{G}^{(\mathbb{T})}_{\tilde{a}_ky}| + C\omega'^2q^{r+1}\sum_{k\in[\![1,\mu]\!]}|\tilde{P}^{(\mathbb{T})}_{\tilde{a}_ky}| \leqslant C\omega'^2|m_{sc}|q^{r+1}.$$

where we bounded $|\tilde{G}_{xl_k}| \leqslant C|m_{sc}|$, and used the estimate (3.168), the bound for Γ and the fact that for all $k \in [\![1, \mu]\!]$ with at most $\omega + 1$ exceptions, $\tilde{P}^{(\mathbb{T})}_{\tilde{a}_ky}$ are zero.

For $x \in \mathbb{T}$ and $y \notin \mathbb{B}_{2r}(1, \tilde{\mathcal{G}})$, similarly, we have:

$$|\tilde{G}_{xy}| \leqslant \frac{1}{\sqrt{d-1}} \sum_{k \in [\![1,\mu]\!]} |\tilde{G}_{xl_k}||\tilde{G}^{(\mathbb{T})}_{\tilde{a}_k y}| \leqslant Cq \sum_{k \in [\![1,\mu]\!]} |\tilde{G}^{(\mathbb{T})}_{\tilde{a}_k y}| \leqslant C\omega'|m_{sc}|q^{r+1},$$

where we bounded $|\tilde{G}_{xl_k}| \leqslant C|m_{sc}|$ and used the estimate (3.169).

For $x, y \in \mathbb{B}_{2r}(1, \tilde{\mathcal{G}})/\mathbb{T}$, we have the Schur complement formula (B.3):

$$\tilde{G} = \tilde{G}^{(\mathbb{T})} + \tilde{G}^{(\mathbb{T})}\tilde{B}\tilde{G}\tilde{B}'\tilde{G}^{(\mathbb{T})},$$

$$\tilde{P} = \tilde{P}^{(\mathbb{T})} + \tilde{P}^{(\mathbb{T})}\tilde{B}\tilde{P}\tilde{B}'\tilde{P}^{(\mathbb{T})}.$$

By taking the difference,

$$\tilde{G} - \tilde{P} = \tilde{G}^{(\mathbb{T})} - \tilde{P}^{(\mathbb{T})} + (\tilde{G}^{(\mathbb{T})} - \tilde{P}^{(\mathbb{T})})\tilde{B}\tilde{G}\tilde{B}'\tilde{G}^{(\mathbb{T})}$$

$$+ \tilde{P}^{(\mathbb{T})}\tilde{B}(\tilde{G} - \tilde{P})\tilde{B}'\tilde{G}^{(\mathbb{T})} + \tilde{P}^{(\mathbb{T})}\tilde{B}\tilde{P}\tilde{B}'(\tilde{G}^{(\mathbb{T})} - \tilde{P}^{(\mathbb{T})}).$$

Notice that $|\tilde{G}^{(\mathbb{T})}_{xy} - \tilde{P}^{(\mathbb{T})}_{xy}| \leqslant C|m_{sc}|q^r$, $|\tilde{G}_{l_k l_m}| \leqslant C|m_{sc}|$, $|\tilde{P}_{l_k l_m}| \leqslant C|m_{sc}|$ and $|\tilde{G}_{l_k l_m} - \tilde{P}_{l_k l_m}| \leqslant C\omega'^2|m_{sc}|q^r$, we have

$$|\tilde{G}_{xy} - \tilde{P}_{xy}| \leqslant C|m_{sc}|q^r + \sum_{k,m \in [\![1,\mu]\!]} |\tilde{G}^{(\mathbb{T})}_{x\tilde{a}_k} - \tilde{P}^{(\mathbb{T})}_{x\tilde{a}_k}|\frac{C|m_{sc}|}{d-1}|\tilde{G}^{(\mathbb{T})}_{\tilde{a}_m y}|$$

(3.187)

$$+ \sum_{k,m \in [\![1,\mu]\!]} |\tilde{P}^{(\mathbb{T})}_{x\tilde{a}_k}|\frac{C\omega'^2|m_{sc}|q^r}{d-1}|\tilde{G}^{(\mathbb{T})}_{\tilde{a}_m y}| + \sum_{k,m \in [\![1,\mu]\!]} |\tilde{P}^{(\mathbb{T})}_{x\tilde{a}_k}|\frac{C|m_{sc}|}{d-1}|\tilde{G}^{(\mathbb{T})}_{\tilde{a}_m y} - \tilde{P}^{(\mathbb{T})}_{\tilde{a}_m y}|.$$

The following estimates follow from Proposition 3.37:

$$\sum_{k \in [\![1,\mu]\!]} |P^{(\mathbb{T})}_{x\tilde{a}_k}|, \sum_{m \in [\![1,\mu]\!]} |\tilde{G}^{(\mathbb{T})}_{\tilde{a}_m y}| \leqslant C|m_{sc}|,$$

$$\sum_{k \in [\![1,\mu]\!]} |\tilde{G}^{(\mathbb{T})}_{x\tilde{a}_k} - P^{(\mathbb{T})}_{x\tilde{a}_k}|, \sum_{m \in [\![1,\mu]\!]} |\tilde{G}^{(\mathbb{T})}_{\tilde{a}_m y} - P^{(\mathbb{T})}_{\tilde{a}_m y}| \leqslant C\omega'|m_{sc}|q^r.$$

Therefore (3.187) simplifies to

$$|\tilde{G}_{xy} - \tilde{P}_{xy}| \leqslant C\omega'^2|m_{sc}|q^r.$$

Finally, for $x \notin \mathbb{B}_{2r}(1, \tilde{\mathcal{G}})$ or $y \notin \mathbb{B}_{2r}(1, \tilde{\mathcal{G}})$, by symmetry we assume $x \notin \mathbb{B}_{2r}(1, \tilde{\mathcal{G}})$, we have

$$
\begin{aligned}
|(\tilde{G}^{(\mathbb{T})} \tilde{B} \tilde{G} \tilde{B}' \tilde{G}^{(\mathbb{T})})_{xy}| &\leqslant \frac{1}{d-1} \sum_{k,m \in [\![1,\mu]\!]} |\tilde{G}^{(\mathbb{T})}_{x\tilde{a}_k}||\tilde{G}_{l_k l_m}||\tilde{G}^{(\mathbb{T})}_{\tilde{a}_m y}| \\
&\leqslant \sum_{k,m \in [\![1,\mu]\!]} |\tilde{G}^{(\mathbb{T})}_{x\tilde{a}_k}| \frac{C|m_{sc}|}{d-1} |\tilde{G}^{(\mathbb{T})}_{\tilde{a}_m y}| \leqslant C\omega'|m_{sc}|q^{r+2},
\end{aligned}
$$

where we used (3.169), thus,

$$
|\tilde{G}_{xy} - \tilde{G}^{(\mathbb{T})}_{xy}| \leqslant C\omega'|m_{sc}|q^{r+2}.
$$

Altogether we proved that

$$
\left| \tilde{G}_{ij} - \tilde{P}_{ij}(\mathcal{E}_r(i,j,\tilde{\mathcal{G}})) \right| \leqslant C|m_{sc}|\omega'^2 q^r \ll q^{\ell+1},
$$

provided that $\omega'^2 q^\ell \ll 1$. The weak stability estimates (3.197) follows by combining with (2.15). This finishes the proof of (3.156). $\qquad \square$

3.9. **Concentration in the switched graph.** The result of this section is the following proposition, which shows that the average of the Green's function of $\tilde{\mathcal{G}}^{(\mathbb{T})}$ over the vertex boundary of $\mathbb{T}$ concentrates under resampling of the edge boundary of $\mathbb{T}$. This part is where the condition that the edge boundary contains $\gg \log N$ edges is important.

More precisely, recall the vertex boundary $\mathbb{I} = \{\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_\mu\}$ of $\mathcal{T}$ in $\tilde{\mathcal{G}}$ from (3.135). For any finite graph $\mathcal{H}$ (not necessarily regular and not necessary on N vertices), we define

$$
(3.188) \qquad Q(\mathcal{H}, z) = \frac{1}{Nd} \sum_{(i,j) \in \vec{E}} G^{(i)}_{jj}(\mathcal{H}, z),
$$

where $\vec{E}$ denotes the set of oriented edges of $\mathcal{H}$, and $G^{(i)}(\mathcal{H}, z)$ the Green's function of the graph obtained from $\mathcal{H}$ by removing the vertex i. Notice that we always normalize

(3.188) by Nd, irregardless of the actual number of oriented edges in $\mathcal{H}$ (which can be smaller than Nd).

Proposition 3.40. *Sets $\Omega_1^+(z, \ell)$ and $\bar{\Omega}$ are as defined in Section 3.1.2. Let $z \in \mathbb{C}_+$ and $\mathcal{G} \in \Omega_1^+(z, \ell)$. Then there exists an event $F'(\mathcal{G}) \subset F(\mathcal{G})$ (as in Section 3.6.1) with probability $\mathbb{P}_{\mathcal{G}}(F'(\mathcal{G})) = 1 - \mathrm{o}(N^{-\omega+\delta})$ such that for any $\mathbf{S} \in F'(\mathcal{G})$ with $T_{\mathbf{S}}(\mathcal{G}) \in \bar{\Omega}$,*

(3.189)
$$\left| \frac{1}{\mu} \sum_{k=1}^{\mu} \left(\tilde{G}^{(\mathbb{T})}_{\tilde{a}_k \tilde{a}_k} - P_{\tilde{a}_k \tilde{a}_k}(\mathcal{E}_r(\tilde{a}_k, \tilde{a}_k, \tilde{\mathcal{G}}^{(\mathbb{T})})) \right) - \left(Q(\tilde{\mathcal{G}}) - m_{sc} \right) \right| \leqslant \frac{2(\log N)^{1/2+\delta} |m_{sc}| q^r}{\sqrt{\mu}},$$

provided that $\sqrt{d-1} \geqslant \max\{(\omega+1)^2 2^{2\omega+10}, 2^8(\omega+1)K\}$, $\omega'^2 q^\ell \ll 1$ and $\sqrt{N}\eta q^{3r+2} \geqslant M$.

To prove Proposition 3.40, in Lemma 3.42, we first show a similar statement for the unswitched graph $\mathcal{G}^{(\mathbb{T})}$ in which the problem becomes a concentration problem of *independent* random variables. Then we prove Proposition 3.40 by comparision, using the estimates of Proposition 3.26, and the fact that the change from $Q(\tilde{\mathcal{G}}, z)$ to $Q(\mathcal{G}^{(\mathbb{T})}, z)$ is small (Lemma 3.43). Proposition 3.26 is applicable since, by the definition of set $\Omega_1^+(z, \ell)$ in Section 3.1, any graph $\mathcal{G} \in \Omega_1^+(z, \ell)$ satisfies the assumptions in Proposition 3.26 with $K = 2^{10}$.

The following proposition is used repeatedly in this section. It follows from exactly the same argument as Proposition 3.22, and we therefore omit the proof.

Lemma 3.41. *Given $z \in \mathbb{C}_+$, a constant $K' \geqslant 2$, and $\mathcal{G} \in \bar{\Omega}$. Let $\mathcal{H}$ be one of the graphs $\mathcal{G}^{(\mathbb{T})}$, $\hat{\mathcal{G}}^{(\mathbb{T})}$, $\tilde{\mathcal{G}}^{(\mathbb{T})}$ or $\tilde{\mathcal{G}}$, and suppose that*

(3.190)
$$|G_{ij}(\mathcal{H}, z) - P_{ij}(\mathcal{E}_r(i, j, \mathcal{H}), z)| \leqslant K'|m_{sc}|q^r.$$

Then, for any vertices i, j in $\mathcal{H}^{(x)}$, we have

(3.191)
$$|G_{ij}(\mathcal{H}^{(x)}, z) - P_{ij}(\mathcal{E}_r(i, j, \mathcal{H}^{(x)}), z)| \leqslant 2K'|m_{sc}|q^r,$$

provided that $\sqrt{d-1} \geqslant (\omega + 1)^2 2^{2\omega + 10}$. *Here all graphs have deficit function* $g = d - \deg$, *and we recall that* $\mathcal{H}^{(x)}$ *is the graph obtained from* $\mathcal{H}$ *by removing the vertex* x.

3.9.1. *Estimate for the unswitched graph.* The next lemma shows concentration of a certain average of the Green's function in the unswitched graph.

Lemma 3.42. *For any* $z \in \mathbb{C}_+$ *and* $\mathcal{G} \in \Omega_1^+(z, \ell)$, *we define the set* $F'(\mathcal{G}) \subset F(\mathcal{G})$ *(as in Section 3.6.1) such that*

(3.192)
$$\left| \frac{1}{\mu} \sum_{k=1}^{\mu} \left(G_{c_k c_k}^{(\mathbb{T} b_k)} - P_{c_k c_k}(\mathcal{E}_r(c_k, c_k, \mathcal{G}^{(\mathbb{T} b_k)})) \right) - \left(Q(\mathcal{G}^{(\mathbb{T})}) - m_{sc} \right) \right| \leqslant \frac{(\log N)^{1/2 + \delta} |m_{sc}| q^r}{\sqrt{\mu}}.$$

Then $\mathbb{P}_{\mathcal{G}}(F'(\mathcal{G})) = 1 - o(N^{-\omega + \delta})$.

Proof. Let

$$X_k = G_{c_k c_k}^{(\mathbb{T} b_k)} - P_{c_k c_k}(\mathcal{E}_r(c_k, c_k, \mathcal{G}^{(\mathbb{T} b_k)})), \quad k \in [\![1, \mu]\!].$$

Conditioned on the graph $\mathcal{G}^{(\mathbb{T})}$, the random sets $\vec{S}_1, \vec{S}_2, \ldots, \vec{S}_\mu$ are independent and identically distributed, and thus $X_1, X_2, \ldots, X_\mu$ are i.i.d random variables. By Lemma 3.41 and the assumption that $\mathcal{G} \in \Omega_1^+(z, \ell)$, for any $k \in [\![1, \mu]\!]$, we have

$$|X_k| \leqslant 2K |m_{sc}| q^r,$$

where $K = 2^{10}$. By Azuma's inequality for independent random variables, it therefore follows that

(3.193)
$$\mathbb{P}_{\mathcal{G}} \left(\left| \frac{1}{\mu} \sum_{k=1}^{\mu} X_k - \mathbb{E}[X_k] \right| \geqslant \frac{2Kt |m_{sc}| q^r}{\sqrt{\mu}} \right) \leqslant e^{-t^2/2}.$$

In the following, we still need to estimate $\mathbb{E}[X_k]$. Let $\vec{E}$ be the set of oriented edges of $\mathcal{G}^{(\mathbb{T})}$. By definition, $\mathbb{T}$ is the ℓ neighborhood of the vertex 1, and by the trivial

bound it intersects at most $d + d(d-1) + \cdots + d(d-1)^\ell \leqslant 2(d-1)^{\ell+1}$ edges. Thus $Nd - 4(d-1)^{\ell+1} \leqslant |\vec{E}| \leqslant Nd$. Using that, by Lemma 3.41, we also have $|G_{ii}^{(\mathbb{T}j)} - P_{ii}(\mathcal{E}_r(i, i, \mathcal{G}^{(\mathbb{T}j)}))| \leqslant 2K|m_{sc}|q^r$, it follows that

$$\mathbb{E}[X_k] = \frac{1}{|\vec{E}|} \sum_{(i,j)\in\vec{E}} G_{ii}^{(\mathbb{T}j)} - P_{ii}(\mathcal{E}_r(i, i, \mathcal{G}^{(\mathbb{T}j)}))$$

$$= \frac{1}{Nd} \sum_{(i,j)\in\vec{E}} G_{ii}^{(\mathbb{T}j)} - P_{ii}(\mathcal{E}_r(i, i, \mathcal{G}^{(\mathbb{T}j)})) + \mathrm{O}_{\leqslant}\left(\frac{8K|m_{sc}|q^r(d-1)^{\ell+1}}{Nd}\right)$$

$$(3.194) \qquad = Q(\mathcal{G}^{(\mathbb{T})}) - \frac{1}{Nd} \sum_{(i,j)\in\vec{E}} P_{ii}(\mathcal{E}_r(i, i, \mathcal{G}^{(\mathbb{T}j)})) + \mathrm{O}_{\leqslant}\left(\frac{8K}{N}\right).$$

Moreover, since by assumption $\mathcal{G} \in \bar{\Omega}$, all except for at most N^δ vertices have radius-R tree neighborhoods in $\mathcal{G}$, and therefore

$$|\{i \in [\![1, N]\!] \setminus \mathbb{T} : \mathcal{B}_r(i, \mathcal{G}^{(\mathbb{T})}) \text{ is not a } d\text{-regular tree}\}|$$

$$\leqslant |\{i \in [\![1, N]\!] : \mathcal{B}_r(i, \mathcal{G}) \text{ is not a tree}\}| + |\{i \in [\![1, N]\!] : \mathrm{dist}_{\mathcal{G}}(i, \mathbb{T}) \leqslant r\}|$$

$$\leqslant N^\delta + 2(d-1)^{r+\ell} \leqslant 2N^\delta.$$

For the vertices i contained in the set on the left-hand side, we have the bound $|P_{ii}(\mathcal{E}_r(i, i, \mathcal{G}^{(\mathbb{T}j)}))| \leqslant 2|m_{sc}|$ from (2.13). For the other vertices i, whose r-neighborhood in $\mathcal{G}^{(\mathbb{T})}$ is a d-regular tree, we have the equality $P_{ii}(\mathcal{E}_r(i, i, \mathcal{G}^{(\mathbb{T}j)})) = m_{sc}$. Therefore

$$(3.195) \qquad \frac{1}{Nd} \sum_{(i,j)\in\vec{E}} P_{ii}(\mathcal{E}_r(i, i, \mathcal{G}^{(\mathbb{T}j)})) = m_{sc} + \mathrm{O}_{\leqslant}(8N^{-1+\delta}).$$

Combining (3.194), (3.195), and taking $t = (\log N)^{1/2+\delta}/(4K)$ in (3.193), we get

$$\mathbb{P}_{\mathcal{G}}\left(\left|\frac{1}{\mu}\sum_{i=1}^{\mu} X_i - \left(Q(\mathcal{G}^{(\mathbb{T})}) - m_{sc}\right)\right| \geqslant \frac{2Kt|m_{sc}|q^r}{\sqrt{\mu}} + \frac{10}{N^{1-\delta}}\right) \leqslant e^{-(\log N)^{1+2\delta}/(32K^2)}.$$

Since $N^{-1+\delta} \ll (\log N)^{1/2+\delta} |m_{sc}| q^r / \sqrt{\mu}$, it follows that (3.192) holds with overwhelming probability, and we can define $F'(\mathcal{G}) \subset F(\mathcal{G})$ as claimed with probability

$$\mathbb{P}_{\mathcal{G}}(F'(\mathcal{G})) \geqslant \mathbb{P}_{\mathcal{G}}(F(\mathcal{G})) - e^{-(\log N)^{1+2\delta}/(32K^2)} = 1 - o(N^{-\omega+\delta}),$$

where we used (3.104). This completes the proof. $\qquad\square$

3.9.2. *Changing $Q(\mathcal{G}^{(\mathbb{T})})$ to $Q(\tilde{\mathcal{G}})$.* The next lemma shows that we can replace $Q(\mathcal{G}^{(\mathbb{T})})$ by $Q(\mathcal{G})$ up to a small error. It follows from the general insensitivity of the quantity Q to small changes of the graph.

Lemma 3.43. *For $z \in \mathbb{C}_+$, $\mathcal{G} \in \Omega_1^+(z, \ell)$ and $\mathbf{S} \in F(\mathcal{G})$ with $T_{\mathbf{S}}(\mathcal{G}) \in \bar{\Omega}$, we have*

$$(3.196) \qquad\qquad |Q(\mathcal{G}^{(\mathbb{T})}, z) - Q(\tilde{\mathcal{G}}, z)| \leqslant \frac{36 d^{2\ell+2}}{N\eta},$$

provided that $\sqrt{d-1} \geqslant \max\{(\omega+1)^2 2^{2\omega+10}, 2^8(\omega+1)K\}$, $\omega'^2 q^\ell \ll 1$ and $\sqrt{N\eta} q^{3r+2} \geqslant M$.

The proof of Lemma 3.43 uses Lemma 3.44 below, which is a direct consequence of the Ward identity (B.6).

Lemma 3.44. *Given a graph $\mathcal{G}$ with degree bounded by d. We denote by $\vec{E}$ the set of oriented edges of $\mathcal{G}$, by H its normalized adjacency matrix, and by $G = (H - z)^{-1}$ its Green's function. Then, if for some $z \in \mathbb{C}_+$ and any $(i, j) \in \vec{E}$, it holds that*

$$(3.197) \qquad\qquad |G_{ij}(z)| \leqslant |G_{jj}(z)| \leqslant 2,$$

then for any vertex $x \in \mathcal{G}$,

$$(3.198) \qquad\qquad \sum_{(i,j)\in\vec{E}} |G_{ix}^{(j)}(z)|^2 \leqslant \frac{8d}{\eta}.$$

Proof. By the Schur complement formula (B.5) and the Ward identity (B.6), we obtain

$$\sum_{(i,j)\in\vec{E}} |G_{ix}^{(j)}|^2 = \sum_{(i,j)\in\vec{E}} \left| G_{ix} - \frac{G_{ij}G_{jx}}{G_{jj}} \right|^2 \leqslant \sum_{(i,j)\in\vec{E}} 2\,|G_{ix}|^2 + 2\left| \frac{G_{ij}G_{jx}}{G_{jj}} \right|^2$$

$$\leqslant 4 \sum_{(i,j)\in\vec{E}} |G_{ix}|^2 \leqslant 4 \sum_i \deg_{\mathcal{G}}(i)|G_{ix}|^2 \leqslant \frac{4d\mathrm{Im}[G_{xx}]}{\eta} \leqslant \frac{8d}{\eta},$$

as claimed. $\qquad\square$

We will prove Lemma 3.43 in two steps, by proving

$$(3.199) \qquad |Q(\mathcal{G}^{(\mathbb{T})}) - Q(\hat{\mathcal{G}}^{(\mathbb{T})})| \leqslant \frac{d\mu}{2N\eta}, \qquad |Q(\hat{\mathcal{G}}^{(\mathbb{T})}) - Q(\tilde{\mathcal{G}}^{(\mathbb{T})})| \leqslant \frac{d\mu}{2N\eta},$$

and

$$(3.200) \qquad |Q(\tilde{\mathcal{G}}^{(\mathbb{T})}) - Q(\tilde{\mathcal{G}})| \leqslant \frac{34d^{2\ell+2}}{N\eta}.$$

Then (3.196) follows by combining (3.199) and (3.200), and using that $\mu \leqslant 2(d-1)^{\ell+1}$. In preparation, we recall from Proposition 3.32 that, for all vertices $i, j \in [\![N]\!]$,

$$(3.201) \qquad |\tilde{G}_{ij}(z)| \leqslant |\tilde{G}_{jj}(z)| \leqslant 2.$$

Proof of (3.199). The proofs of both estimates in (3.199) are analogous, and we only prove the first one. Denote by $\vec{E}$ the set of oriented edges of $\hat{\mathcal{G}}^{(\mathbb{T})}$, and by $\Delta = \sum_{k=1}^{\nu}(e_{b_k c_k} + e_{c_k b_k})/\sqrt{d-1}$ the difference of the normalized adjacency matrices of the graphs $\hat{\mathcal{G}}^{(\mathbb{T})}$ and $\mathcal{G}^{(\mathbb{T})}$. Then by the resolvent formula (B.1),

$$\sum_{(i,j)\in\vec{E}} |\hat{G}_{ii}^{(\mathbb{T}j)} - G_{ii}^{(\mathbb{T}j)}| \leqslant \sum_{x,y} \sum_{(i,j)\in\vec{E}} |\hat{G}_{ix}^{(\mathbb{T}j)}\Delta_{xy}G_{yi}^{(\mathbb{T}j)}|$$

$$\leqslant \sum_{x,y} \Delta_{xy} \left(\sum_{(i,j)\in\vec{E}} |\hat{G}_{ix}^{(\mathbb{T}j)}|^2 \sum_{(i,j)\in\vec{E}} |G_{yi}^{(\mathbb{T}j)}|^2 \right)^{1/2} \leqslant \frac{16d\mu}{\eta\sqrt{d-1}},$$

where we used (3.198) (and that both graphs $\mathcal{G}^{(\mathbb{T})}$ and $\hat{\mathcal{G}}^{(\mathbb{T})}$ satisfy condition (3.197) by the definition of $\Omega_1^+(z, \ell)$ and (3.100)). Therefore,

(3.202)

$$\left| Q(\mathcal{G}^{(\mathbb{T})}) - Q(\hat{\mathcal{G}}^{(\mathbb{T})}) \right| \leqslant \frac{1}{Nd} \sum_{k \in [\![1,\nu]\!]} |G_{b_k b_k}^{(\mathbb{T}c_k)} + G_{c_k c_k}^{(\mathbb{T}b_k)}| + \frac{1}{Nd} \sum_{(i,j) \in \vec{E}} |\hat{G}_{ii}^{(\mathbb{T}j)} - G_{ii}^{(\mathbb{T}j)}|$$

$$\leqslant \frac{4\nu |m_{sc}|}{Nd} + \frac{16d\mu}{N\eta\sqrt{d-1}} \leqslant \frac{d\mu}{2N\eta},$$

where in the estimate of the first term, we used $|G_{b_k b_k}^{(\mathbb{T}c_k)}|, |G_{c_k c_k}^{(\mathbb{T}b_k)}| \leqslant 2|m_{sc}|$ which follows from combining (3.11), Lemma 3.41, and (2.13). $\qquad\square$

Proof of (3.200). The normalized adjacency matrices of $\tilde{\mathcal{G}}$ takes the block form

$$\begin{bmatrix} H & \tilde{B}' \\ \tilde{B} & D \end{bmatrix},$$

where H is the normalized adjacency matrix for $\mathcal{T}$, and $\tilde{B}$ corresponds to the edges from $\mathbb{I}$ to $\mathbb{T}_\ell$, where $\mathbb{I}$ is the set of boundary vertices of $\mathcal{T}$ in the switched graph $\tilde{\mathcal{G}}$ as defined in (3.135). We denote by $\vec{E}$ the set of oriented edges of $\tilde{\mathcal{G}}^{(\mathbb{T})}$. By the Schur complement formula (B.3), we have

$$\sum_{(i,j) \in \vec{E}} |\tilde{G}_{ii}^{(\mathbb{T}j)} - \tilde{G}_{ii}^{(j)}| \leqslant \frac{1}{d-1} \sum_{k,m \in [\![1,\mu]\!]} \sum_{(i,j) \in \vec{E}} |\tilde{G}_{i\tilde{a}_k}^{(\mathbb{T}j)} \tilde{G}_{l_k l_m}^{(j)} \tilde{G}_{\tilde{a}_m i}^{(\mathbb{T}j)}|.$$

It follows from (3.201) and (B.5) that $|\tilde{G}^{(j)}_{l_k l_m}| \leqslant 4$. Therefore the above expression is bounded by

$$\sum_{(i,j)\in\vec{E}} |\tilde{G}^{(\mathbb{T}j)}_{ii} - \tilde{G}^{(j)}_{ii}| \leqslant \frac{4}{d-1} \sum_{k,m\in[\![1,\mu]\!]} \sum_{(i,j)\in\vec{E}} |\tilde{G}^{(\mathbb{T}j)}_{i\tilde{a}_k}\tilde{G}^{(\mathbb{T}j)}_{\tilde{a}_m i}|$$

$$\leqslant \frac{4}{d-1} \sum_{k,m\in[\![1,\mu]\!]} \left(\sum_{(i,j)\in\vec{E}} |\tilde{G}^{(\mathbb{T}j)}_{i\tilde{a}_k}|^2 \sum_{(i,j)\in\vec{E}} |\tilde{G}^{(\mathbb{T}j)}_{\tilde{a}_m i}|^2\right)^{1/2}$$

$$\leqslant \frac{32d\mu^2}{\eta(d-1)} \leqslant \frac{32d^{2\ell+2}}{\eta},$$

where we used (3.198) (since $\tilde{\mathcal{G}}^{(\mathbb{T})}$ satisfies condition (3.197) thanks to the definition of $\Omega^+_1(z,\ell)$ and (3.102)). Therefore, we have

(3.203)
$$\left|Q(\tilde{\mathcal{G}}^{(\mathbb{T})}) - Q(\tilde{\mathcal{G}})\right| \leqslant \frac{1}{Nd} \sum_{\{i,j\} \text{ incident to } \mathbb{T}} |\tilde{G}^{(j)}_{ii} + \tilde{G}^{(i)}_{jj}| + \frac{1}{Nd} \sum_{(i,j)\in\vec{E}} |\tilde{G}^{(\mathbb{T}j)}_{ii} - G^{(j)}_{ii}|$$

$$\leqslant \frac{16(d-1)^{\ell+1}}{Nd} + \frac{32d^{2\ell+2}}{N\eta} \leqslant \frac{34d^{2\ell+2}}{N\eta},$$

where for the first term we used $|\tilde{G}^{(j)}_{ii}|, |\tilde{G}^{(i)}_{jj}| \leqslant 4$ from (3.201) and (B.5) . $\qquad\square$

3.9.3. *Adding of switched vertices.* Recall the index set $J \subset [\![1,\nu]\!]$ from Proposition 3.29. In this subsection, we show that the following lemma.

Lemma 3.45. *For $z \in \mathbb{C}_+$, $\mathcal{G} \in \Omega^+_1(z,\ell)$ and $\mathbf{S} \in F(\mathcal{G})$ with $T_{\mathbf{S}}(\mathcal{G}) \in \bar{\Omega}$, for any $k \in J$, we have*

(3.204)
$$|\hat{G}^{(\mathbb{T}b_k)}_{c_k c_k} - G^{(\mathbb{T}b_k)}_{c_k c_k}| \leqslant 16q^{2r}$$

and

(3.205)
$$|\tilde{G}^{(\mathbb{T})}_{c_k c_k} - \hat{G}^{(\mathbb{T}b_k)}_{c_k c_k}| \leqslant 2^{10} K^4 |m_{sc}| q^{2r},$$

where $K = 2^{10}$. For both estimates, we assume $\sqrt{d-1} \geqslant \max\{(\omega+1)^2 2^{2\omega+10}, 2^8(\omega+1)K\}$, $\omega' q^r \ll 1$ and $\sqrt{N\eta} q^{2r} \geqslant M$.

To prove Lemma 3.45 we need the estimates summarized in the following lemma.

Lemma 3.46. *Let $z \in \mathbb{C}_+$, $\mathcal{G} \in \Omega_1^+(z, \ell)$, and $\mathbf{S} \in F(\mathcal{G})$ with $T_{\mathbf{S}}(\mathcal{G}) \in \bar{\Omega}$. Then for any index $k \in J$, the vertex c_k is far away from $\{a_1, \ldots, a_\mu, b_1, \ldots, b_\nu\}$:*

(3.206)
$$\mathrm{dist}_{\hat{\mathcal{G}}^{(\mathbb{T})}}(c_k, \{a_1, \ldots, a_\mu, b_1, \ldots, b_\nu\}) \geqslant \mathrm{dist}_{\tilde{\mathcal{G}}^{(\mathbb{T})}}(c_k, \{a_1, \ldots, a_\mu, b_1, \ldots, b_\nu\}) > 2r.$$

Moreover, for any $x \in \{a_1, \ldots, a_\mu, b_1, \ldots, b_\nu\}$,

(3.207)
$$|\hat{G}_{c_k x}^{(\mathbb{T})}| \leqslant 2K|m_{sc}|q^r, \quad |\tilde{G}_{c_k x}^{(\mathbb{T})}| \leqslant 2^7 K^3 |m_{sc}|q^r, \quad |\hat{G}_{b_k b_k}^{(\mathbb{T})}| \geqslant |m_{sc}|/2,$$

where $K = 2^{10}$ and we assume that $\sqrt{d-1} \geqslant \max\{(\omega+1)^2 2^{2\omega+10}, 2^8(\omega+1)K\}$, $\omega' q^r \ll 1$ and $\sqrt{N\eta} \geqslant M(d-1)^{\ell+1}$.

Proof. (3.206) is (3.144). The first two estimates in (3.207) follow from (3.206) and $K = 2^{10}$ in Proposition 3.26. The last estimate in (3.207) follows by taking $K = 2^{10}$ in Proposition 3.26 and (2.13). □

Proof of Lemma 3.45. Notice that for $\mathcal{G} \in \Omega_1^+(z, \ell)$, the assumptions in Proposition 3.26 hold for $K = 2^{10}$. By the resolvent identity (B.1),

$$|\hat{G}_{c_k c_k}^{(\mathbb{T} b_k)} - G_{c_k c_k}^{(\mathbb{T} b_k)}| \leqslant \sum_{x,y} |\hat{G}_{c_k x}^{(\mathbb{T} b_k)}||\Delta_{xy}||\hat{G}_{y c_k}^{(\mathbb{T} b_k)}|,$$

where $\Delta = \sum_{m \in [\![1, \nu]\!] \setminus \{k\}} (e_{c_m b_m} + e_{b_m c_m})/\sqrt{d-1}$. By our choice of the index set J, $\{b_k, c_k\}$ and $\{b_m, c_m : m \in [\![1, \nu]\!] \setminus \{k\}\}$ are in different S-cells. Thus $|\hat{G}_{c_k x}^{(\mathbb{T})}|, |\hat{G}_{b_k x}^{(\mathbb{T})}| \leqslant 2M/\sqrt{N\eta}$ by (3.101). Therefore, using (B.5), and notice $|\hat{G}_{b_k b_k}^{(\mathbb{T})}| \geqslant |\hat{G}_{c_k b_k}^{(\mathbb{T})}|$, we get

$$|\hat{G}_{c_k x}^{(\mathbb{T} b_k)}| \leqslant |\hat{G}_{c_k x}^{(\mathbb{T})}| + \left| \frac{\hat{G}_{c_k b_k}^{(\mathbb{T})} \hat{G}_{b_k x}^{(\mathbb{T})}}{\hat{G}_{b_k b_k}^{(\mathbb{T})}} \right| \leqslant |\hat{G}_{c_k x}^{(\mathbb{T})}| + |\hat{G}_{b_k x}^{(\mathbb{T})}| \leqslant \frac{4M}{\sqrt{N\eta}}.$$

The same estimate holds for $|\hat{G}^{(\mathbb{T})}_{yc_k}|$. Thus the second term is bounded by

$$\sum_{x,y} |\hat{G}^{(\mathbb{T}b_k)}_{c_k x}||\Delta_{xy}||\hat{G}^{(\mathbb{T}b_k)}_{yc_k}| \leqslant 4(d-1)^{\ell+1/2}\left(\frac{2M}{\sqrt{N\eta}}\right)^2 \leqslant 16q^{2r},$$

provided that $\sqrt{N\eta} \geqslant Mq^{-3r/2}$. Similarly, now setting $\Delta = \sum_{m=1}^{\nu}(e_{a_m b_m}+e_{b_m a_m})/\sqrt{d-1}$, by the resolvent identity (B.1) and (B.5), we have

(3.208)

$$|\tilde{G}^{(\mathbb{T})}_{c_k c_k} - \hat{G}^{(\mathbb{T}b_k)}_{c_k c_k}| \leqslant |\tilde{G}^{(\mathbb{T})}_{c_k c_k} - \hat{G}^{(\mathbb{T})}_{c_k c_k}| + |\hat{G}^{(\mathbb{T}b_k)}_{c_k c_k} - \hat{G}^{(\mathbb{T})}_{c_k c_k}| \leqslant \sum_{x,y} |\hat{G}^{(\mathbb{T})}_{c_k x}||\Delta_{xy}||\tilde{G}^{(\mathbb{T})}_{yc_k}| + \left|\frac{\hat{G}^{(\mathbb{T})}_{c_k b_k}\hat{G}^{(\mathbb{T})}_{b_k c_k}}{\hat{G}^{(\mathbb{T})}_{b_k b_k}}\right|.$$

For the last term in (3.208), by (3.207), $|\hat{G}^{(\mathbb{T})}_{c_k b_k}| \leqslant 2K|m_{sc}|q^r$ and $|\hat{G}^{(\mathbb{T})}_{b_k b_k}| \geqslant |m_{sc}|/2$, and therefore

$$\left|\frac{\hat{G}^{(\mathbb{T})}_{c_k b_k}\hat{G}^{(\mathbb{T})}_{b_k c_k}}{\hat{G}^{(\mathbb{T})}_{b_k b_k}}\right| \leqslant \frac{(2K|m_{sc}|q^r)^2}{|m_{sc}|/2} = 8K^2|m_{sc}|q^{2r}.$$

For the sum on the right-hand side of (3.208), we can split it into two,

$$\sum_{x,y} |\hat{G}^{(\mathbb{T})}_{c_k x}||\Delta_{xy}||\tilde{G}^{(\mathbb{T})}_{yc_k}| = |\hat{G}^{(\mathbb{T})}_{c_k b_k}||\Delta_{b_k a_k}||\tilde{G}^{(\mathbb{T})}_{a_k c_k}| + \sum_{(x,y)\neq(b_k,a_k)} |\hat{G}^{(\mathbb{T})}_{c_k x}||\Delta_{xy}||\tilde{G}^{(\mathbb{T})}_{yc_k}|.$$

Again, we have $|\hat{G}^{(\mathbb{T})}_{c_k x}| \leqslant 2M/\sqrt{N\eta}$, for $x \in \{b_m : m \in [\![1,\nu]\!]\setminus\{k\}\}\cup\{a_m : m \in [\![1,\nu]\!]\}$. Combining with (3.207), it follows that

$$(3.205) \leqslant \frac{(2K|m_{sc}|q^r)(2^7 K^3|m_{sc}|q^r)}{\sqrt{d-1}} + \frac{2M}{\sqrt{N\eta}}\frac{4(d-1)^{\ell+1}}{\sqrt{d-1}}(2^7 K^3|m_{sc}|q^r) + 8K^2|m_{sc}|q^{2r}$$

$$\leqslant 2^{10}K^4|m_{sc}|q^{2r},$$

provided that $\sqrt{N\eta}q^{2r} \geqslant M$. $\qquad\qquad\square$

3.9.4. *Proof of Proposition 3.40.* Finally, using the previous lemmas, we can proof Proposition 3.40.

Proof of Proposition 3.40. For $k \in J$, the r-neighborhood of c_k is a d-regular tree with root degree $d-1$ in any of the graphs $\mathcal{G}^{(\mathbb{T}b_k)}$, $\hat{\mathcal{G}}^{(\mathbb{T}b_k)}$ and $\tilde{\mathcal{G}}^{(\mathbb{T})}$; therefore, by (2.11),

$$P_{c_k c_k} \left(\mathcal{E}_r \left(c_k, c_k, \mathcal{G}^{(\mathbb{T}b_k)} \right) \right) = P_{c_k c_k} \left(\mathcal{E}_r \left(c_k, c_k, \hat{\mathcal{G}}^{(\mathbb{T}b_k)} \right) \right) = P_{c_k c_k} \left(\mathcal{E}_r \left(c_k, c_k, \tilde{\mathcal{G}}^{(\mathbb{T})} \right) \right) = m_{sc}.$$

On the other hand, for the indices $k \in [\![1, \mu]\!] \setminus J$, by (3.11), Proposition 3.26, and Lemma 3.41, using that for $\mathcal{G} \in \Omega_1^+(z, \ell)$, the assumption of Proposition 3.26 holds with $K = 2^{10}$, we have

$$\left| G_{c_k c_k}^{(\mathbb{T}b_k)} - P_{c_k c_k} \left(\mathcal{E}_r \left(c_k, c_k, \mathcal{G}^{(\mathbb{T}b_k)} \right) \right) \right| \leqslant 2K |m_{sc}| q^r,$$

(3.209) $$\left| \hat{G}_{c_k c_k}^{(\mathbb{T}b_k)} - P_{c_k c_k} \left(\mathcal{E}_r \left(c_k, c_k, \hat{\mathcal{G}}^{(\mathbb{T}b_k)} \right) \right) \right| \leqslant 4K |m_{sc}| q^r,$$

$$\left| \tilde{G}_{\tilde{a}_k \tilde{a}_k}^{(\mathbb{T})} - P_{\tilde{a}_k \tilde{a}_k} \left(\mathcal{E}_r \left(\tilde{a}_k, \tilde{a}_k, \tilde{\mathcal{G}}^{(\mathbb{T})} \right) \right) \right| \leqslant 2^7 K^3 |m_{sc}| q^r.$$

The above estimates (3.209) and (3.204) give

(3.210)

$$\left| \frac{1}{\mu} \sum_{k=1}^{\mu} \left(G_{c_k c_k}^{(\mathbb{T}b_k)} - P_{c_k c_k} \left(\mathcal{E}_r \left(c_k, c_k, \mathcal{G}^{(\mathbb{T}b_k)} \right) \right) \right) - \left(\hat{G}_{c_k c_k}^{(\mathbb{T}b_k)} - P_{c_k c_k} \left(\mathcal{E}_r \left(c_k, c_k, \hat{\mathcal{G}}^{(\mathbb{T}b_k)} \right) \right) \right) \right|$$

$$\leqslant \frac{6K(\mu - |J|)|m_{sc}| q^r}{\mu} + \frac{1}{\mu} \sum_{k \in J} |G_{c_k c_k}^{(\mathbb{T}b_k)} - \hat{G}_{c_k c_k}^{(\mathbb{T}b_k)}|$$

$$\leqslant \frac{6K(\omega' + 9\omega)|m_{sc}| q^r}{\mu} + (8K^2 |m_{sc}| q^{2r} + 16 q^{2r}) \leqslant \frac{(\log N)^{1/2+\delta} |m_{sc}| q^r}{4\sqrt{\mu}}.$$

Moreover, by the above estimates (3.209), (3.205) and using $\tilde{a}_k = c_k$ for $k \in J$, we have

(3.211)

$$
\left| \frac{1}{\mu} \sum_{k=1}^{\mu} \left(\tilde{G}_{\tilde{a}_k \tilde{a}_k}^{(\mathbb{T})} - P_{\tilde{a}_k \tilde{a}_k} \left(\mathcal{E}_r \left(\tilde{a}_k, \tilde{a}_k, \tilde{\mathcal{G}}^{(\mathbb{T})} \right) \right) \right) - \left(\hat{G}_{c_k c_k}^{(\mathbb{T} b_k)} - P_{c_k c_k} \left(\mathcal{E}_r \left(c_k, c_k, \hat{\mathcal{G}}^{(\mathbb{T} b_k)} \right) \right) \right) \right|
$$

$$
\leqslant \frac{(4K + 2^7 K^3)(\mu - |J|)|m_{sc}|q^r}{\mu} + \frac{1}{\mu} \sum_{k \in J} |\tilde{G}_{c_k c_k}^{(\mathbb{T})} - \hat{G}_{c_k c_k}^{(\mathbb{T} b_k)}|
$$

$$
\leqslant \frac{(4K + 2^7 K^3)(\omega' + 9\omega)|m_{sc}|q^r}{\mu} + 2^{10} K^4 |m_{sc}|q^{2r} \leqslant \frac{(\log N)^{1/2+\delta}|m_{sc}|q^r}{4\sqrt{\mu}}.
$$

In the above estimates we used $\ell \geqslant 4 \log_{d-1} \log N$ by (3.5) so that $\sqrt{\mu} \gg \log N = \omega'$. The left-hand side of (3.189) is bounded by

$$
|(3.192)| + |(3.196)| + |(3.211)| + |(3.192)| \leqslant \frac{2(\log N)^{1/2+\delta}|m_{sc}|q^r}{\sqrt{\mu}},
$$

provided that $\sqrt{N\eta}q^{3r+2} \geqslant M$. $\qquad\square$

3.10. **Improved approximation in the switched graph.** The results of this section are the following proposition, stating that the Green's function obeys better estimates than the original one near vertex 1. As in the previous sections, we write $\tilde{\mathcal{G}} = T_{\mathbf{S}}(\mathcal{G})$ and assume that $\mathbf{S} \in F'(\mathcal{G})$ (as in Lemma 3.42) is such that $\tilde{\mathcal{G}} = T_{\mathbf{S}}(\mathcal{G}) \in \bar{\Omega}$ (as in Section 3.1.2). Throughout the proof, C represents constants depending only on the constant K from (3.99) and the excess ω, which may be different from line to line.

Proposition 3.47. *Under the assumptions of Propositions 3.26, for $\mathbf{S} \in F'(\mathcal{G})$ such that $\tilde{\mathcal{G}} = T_{\mathbf{S}}\mathcal{G} \in \bar{\Omega}$, the Green's function of the switched graph satisfies the following improved estimates near vertex 1.*

(i) *For the vertex $x = 1$,*

(3.212)

$$\tilde{G}_{11} = P_{11}(\mathcal{E}_r(1,1,\tilde{\mathcal{G}})) + \frac{m_d^2 m_{sc}^{2\ell}\mu}{(d-1)^{(\ell+1)}}(Q(\tilde{\mathcal{G}}) - m_{sc}) + O_{\leqslant}\left(2^{2\omega+10}K^3|m_{sc}|q^{r+1}\right).$$

(ii) *For all vertices $x \in [\![2, N]\!]$,*

(3.213) $$\left|\tilde{G}_{1x} - P_{1x}(\mathcal{E}_r(1,x,\tilde{\mathcal{G}}))\right| \leqslant (\omega+1)2^{2\omega+14}K^3|m_{sc}|q^{r+1}.$$

Moreover, if the vertex 1 has radius-R tree neighborhood in the graph $\tilde{\mathcal{G}}$, then the following stronger estimates hold.

(i') *For the vertex $x = 1$,*

(3.214) $$\tilde{G}_{11} = m_d + m_d^2 m_{sc}^{2\ell}\frac{d}{d-1}(Q(\tilde{\mathcal{G}}) - m_{sc}) + O_{\leqslant}\left(\frac{4(\log N)^{1/2+\delta}|m_{sc}|q^r}{\sqrt{d(d-1)^\ell}}\right).$$

(ii') *For the the average of $\tilde{G}_{1x}$ over the vertices x adjacent to 1,*

(3.215)

$$\frac{1}{d}\sum_{1\sim x}\tilde{G}_{1x} + \frac{m_d m_{sc}}{\sqrt{d-1}} = \frac{-m_d^2 m_{sc}^{2\ell-1}(1+m_{sc}^2)}{\sqrt{d-1}}(Q(\tilde{\mathcal{G}}) - m_{sc}) + O_{\leqslant}\left(\frac{16(\log N)^{1/2+\delta}|m_{sc}|q^{r+1}}{\sqrt{d(d-1)^\ell}}\right).$$

For all estimates we assume that $\sqrt{d-1} \geqslant \max\{(\omega+1)^2 2^{2\omega+10}, 2^8(\omega+1)K\}$, $\omega'^2 q^\ell \ll 1$ and $\sqrt{N}\eta q^{3r+2} \geqslant M$, and the global quantity $Q(\tilde{\mathcal{G}})$ is as defined in (3.188).

We use the same set-up as in Section 3.8, and notice that (3.213) is (3.157).

3.10.1. *Proof of* (3.212) *and* (3.214). By (3.163), we have

(3.216)

$$\tilde{G}_{11} - \tilde{P}_{11} = \frac{1}{d-1}\sum_{k\in[\![1,\mu]\!]}\tilde{P}_{1l_k}^2(\tilde{G}_{\tilde{a}_k\tilde{a}_k}^{(\mathbb{T})} - \tilde{P}_{\tilde{a}_k\tilde{a}_k}^{(\mathbb{T})}) + \frac{1}{d-1}\sum_{k\neq m\in[\![1,\mu]\!]}\tilde{G}_{1l_k}\tilde{P}_{1l_m}(\tilde{G}_{\tilde{a}_k\tilde{a}_m}^{(\mathbb{T})} - \tilde{P}_{\tilde{a}_k\tilde{a}_m}^{(\mathbb{T})})$$

$$+ \frac{1}{d-1}\sum_{k\in[\![1,\mu]\!]}(\tilde{G}_{1l_k} - \tilde{P}_{1l_k})\tilde{P}_{1l_k}(\tilde{G}_{\tilde{a}_k\tilde{a}_k}^{(\mathbb{T})} - \tilde{P}_{\tilde{a}_k\tilde{a}_k}^{(\mathbb{T})}).$$

For the last term on the right-hand side of (3.216), we have

$$\left| \frac{1}{d-1} \sum_{k \in [\![1,\mu]\!]} (\tilde{G}_{1l_k} - \tilde{P}_{1l_k}) \tilde{P}_{1l_k} (\tilde{G}^{(\mathbb{T})}_{\tilde{a}_k \tilde{a}_k} - \tilde{P}^{(\mathbb{T})}_{\tilde{a}_k \tilde{a}_k}) \right| \leqslant \sum_{k \in [\![1,\mu]\!]} C q^{r+2} q^{\ell+1} (|m_{sc}|q^r) \leqslant C |m_{sc}| q^{r+\ell+2},$$

where we used (3.177) for the first factor, (3.180) for the second factor, and (3.102) for the last factor. For the second term on the right-hand side of (3.216), we have

$$\left| \frac{1}{d-1} \sum_{k \neq m \in [\![1,\mu]\!]} \tilde{G}_{1l_k} \tilde{P}_{1l_m} (\tilde{G}^{(\mathbb{T})}_{\tilde{a}_k \tilde{a}_m} - \tilde{P}^{(\mathbb{T})}_{\tilde{a}_k \tilde{a}_m}) \right| \leqslant C(q^{\ell+1})(q^{\ell+1}) \sum_{k \neq m \in [\![1,\mu]\!]} |(\tilde{G}^{(\mathbb{T})}_{\tilde{a}_k \tilde{a}_m} - \tilde{P}^{(\mathbb{T})}_{\tilde{a}_k \tilde{a}_m})|$$

$$\leqslant C q^{2\ell+2} (\omega'^2 |m_{sc}|q^r) \leqslant C |m_{sc}| q^{r+\ell+2},$$

provided that $\omega'^2 q^\ell \ll 1$, where we used (3.184) for the first factor, (3.180) for the second factor, and (3.170) for the last factor. Therefore (3.216) is bounded by

$$(3.217) \qquad \tilde{G}_{11} - \tilde{P}_{11} = \frac{1}{d-1} \sum_{k \in [\![1,\mu]\!]} \tilde{P}^2_{1l_k} (\tilde{G}^{(\mathbb{T})}_{\tilde{a}_k \tilde{a}_k} - \tilde{P}^{(\mathbb{T})}_{\tilde{a}_k \tilde{a}_k}) + O\left(|m_{sc}| q^{r+\ell+2} \right),$$

where the implicit constant depends only on the excess ω and K from (3.99).

Proof of (3.214). If the radius-R neighborhood of the vertex 1 is a tree, then by Proposition 2.6,

$$\tilde{P}^2_{1l_k} = \frac{m_d^2 m_{sc}^{2\ell}}{(d-1)^\ell}, \quad \tilde{P}_{11} = m_d,$$

and

$$(3.218) \qquad \tilde{G}_{11} - m_d = \frac{m_d^2 m_{sc}^{2\ell}}{(d-1)^{\ell+1}} \sum_{k \in [\![1,\mu]\!]} (\tilde{G}^{(\mathbb{T})}_{\tilde{a}_k \tilde{a}_k} - \tilde{P}^{(\mathbb{T})}_{\tilde{a}_k \tilde{a}_k}) + O\left(|m_{sc}| q^{r+\ell+2} \right).$$

Notice that $\mu = d(d-1)^\ell$ under the assumption that the R-neighborhood is a tree. Moreover, for all $k \in [\![1, \mu]\!]$,

$$\tilde{P}^{(\mathbb{T})}_{\tilde{a}_k \tilde{a}_k} = \tilde{P}_{\tilde{a}_k \tilde{a}_k} \left(\mathcal{E}_r \left(\tilde{a}_k, \tilde{a}_k, \tilde{\mathcal{G}}^{(\mathbb{T})} \right) \right) = m_{sc},$$

and by Proposition 3.40, we can simplify (3.218) to get

$$(3.219) \qquad \tilde{G}_{11} = m_d + \frac{d}{d-1} m_d^2 m_{sc}^{2\ell}(Q(\tilde{\mathcal{G}}) - m_{sc}) + \mathrm{O}_{\leqslant} \left(\frac{4(\log N)^{1/2+\delta}|m_{sc}|q^r}{\sqrt{d(d-1)^\ell}} \right).$$

This finishes the proof of (3.214). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Proof of (3.212). Since by assumption $\tilde{\mathcal{G}} \in \bar{\Omega}$, the radius-$R$ neighborhood of the vertex 1 has excess at most ω. Therefore, there are at most $2\omega(d-1)^\ell$ indices $k \in [\![1, \mu]\!]$ such that the non-backtracking path from 1 to l_k of length ℓ is not unique. Let

$$J' = \{k \in [\![1, \mu]\!] : \text{non-backtracking path from 1 to } l_k \text{ of length } \ell \text{ is unique}\}.$$

For $k \in J'$, by (2.23) in the proof of Proposition 2.7, we have

$$\left| \tilde{P}_{1l_k} - \frac{m_d(-m_{sc})^\ell}{(d-1)^{\ell/2}} \right| \leqslant |m_d| \sum_{k \geqslant 2} 2^{\omega k} q^{\ell+k-1} \leqslant 2^{2\omega}|m_d|\frac{3}{2}q^{\ell+1},$$

provided that $\sqrt{d-1} \geqslant 2^{\omega+2}$. Therefore, for all $k \in J'$, the following estimate holds

$$\frac{\tilde{P}_{1l_k}^2}{d-1} = \frac{m_d^2 m_{sc}^{2\ell}}{(d-1)^{\ell+1}} + \mathrm{O}_{\leqslant}\left(2^{2\omega+2}q^{2\ell+3}\right),$$

For $k \in [\![1, \mu]\!] \setminus J'$ by (3.180), we have $|\tilde{P}_{1l_k}| \leqslant 2^{\omega+2}|m_{sc}|q^\ell$. Notice that $|J'| \leqslant \mu \leqslant d(d-1)^\ell$ and $|[\![1, \mu]\!] \setminus J'| \leqslant 2\omega(d-1)^\ell$, it follows that

$$(3.220) \quad \begin{aligned} \frac{1}{d-1}\sum_{k \in [\![1,\mu]\!]} &\left| \tilde{P}_{1l_k}^2 - \frac{m_d^2 m_{sc}^{2\ell}}{(d-1)^\ell} \right| = \frac{1}{d-1}\sum_{k \in [\![1,\mu]\!] \setminus J'}(\cdots) + \frac{1}{d-1}\sum_{k \in J'}(\cdots). \\ &\leqslant 2\omega(d-1)^\ell 2^{2\omega+5}q^{2\ell+2} + d(d-1)^\ell 2^{2\omega+2}q^{2\ell+3} \leqslant 2^{2\omega+2}(dq+16\omega)q^2 \end{aligned}$$

Combining (3.220), (3.102) and (3.160), (3.217) leads to

$$\tilde{G}_{11} - \tilde{P}_{11} = \frac{m_d^2 m_{sc}^{2\ell}}{(d-1)^{(\ell+1)}} \sum_{k \in [\![1,\mu]\!]} (\tilde{G}_{\tilde{a}_k \tilde{a}_k}^{(\mathbb{T})} - \tilde{P}_{\tilde{a}_k \tilde{a}_k}^{(\mathbb{T})})$$

(3.221)
$$+ \mathrm{O}_{\leqslant} \left(2^{2\omega+2}(dq + 16\omega)q^2 \max_{k \in [\![1,\mu]\!]} |\tilde{G}_{\tilde{a}_k \tilde{a}_k}^{(\mathbb{T})} - \tilde{P}_{\tilde{a}_k \tilde{a}_k}^{(\mathbb{T})}| \right)$$

$$= \frac{m_d^2 m_{sc}^{2\ell}}{(d-1)^{(\ell+1)}} \sum_{k \in [\![1,\mu]\!]} \left(\tilde{G}_{\tilde{a}_k \tilde{a}_k}^{(\mathbb{T})} - \tilde{P}_{\tilde{a}_k \tilde{a}_k}(\mathcal{E}_r(\tilde{a}_k, \tilde{a}_k, \tilde{\mathcal{G}}^{(\mathbb{T})})) \right) + \mathcal{E},$$

where the error term is bounded

$$|\mathcal{E}| \leqslant 2^{2\omega+2}(dq + 16\omega)q^2(2^7 K^3 |m_{sc}|q^r + 2^{2\omega+3}|m_{sc}|q^{r+1}) + |m_d|^2 \mu 2^{2\omega+3}|m_{sc}|q^{r+1}/(d-1)^{\ell+1}$$

$$\leqslant 3 \times 2^{2\omega+8} K^3 |m_{sc}|q^{r+1},$$

provided that $\sqrt{d-1} \geqslant 2^6\omega$. Therefore, by Proposition 3.40, we can simplify (3.221) to get

(3.222)
$$\tilde{G}_{11} = \tilde{P}_{11}(\mathcal{E}_r(1,1,\tilde{\mathcal{G}})) + \frac{m_d^2 m_{sc}^{2\ell} \mu}{(d-1)^{(\ell+1)}}(Q(\tilde{\mathcal{G}}) - m_{sc}) + \mathrm{O}_{\leqslant}\left(2^{2\omega+10} K^3 |m_{sc}|q^{r+1} \right).$$

This finishes the proof of (3.212). □

3.10.2. *Proof of* (3.215).

Proof of (3.215). For any vertex x adjacent to 1, by (3.163) we have,

(3.223)
$$\tilde{G}_{1x} - \tilde{P}_{1x} = \frac{1}{d-1} \sum_{k \in [\![1,\mu]\!]} \tilde{P}_{1l_k} \tilde{P}_{xl_k}(\tilde{G}_{\tilde{a}_k \tilde{a}_k}^{(\mathbb{T})} - \tilde{P}_{\tilde{a}_k \tilde{a}_k}^{(\mathbb{T})}) + \frac{1}{d-1} \sum_{k \neq m \in [\![1,\mu]\!]} \tilde{G}_{1l_k} \tilde{P}_{xl_m}(\tilde{G}_{\tilde{a}_k \tilde{a}_m}^{(\mathbb{T})} - \tilde{P}_{\tilde{a}_k \tilde{a}_m}^{(\mathbb{T})})$$

$$+ \frac{1}{d-1} \sum_{k \in [\![1,\mu]\!]} (\tilde{G}_{1l_k} - \tilde{P}_{1l_k})\tilde{P}_{xl_k}(\tilde{G}_{\tilde{a}_k \tilde{a}_k}^{(\mathbb{T})} - \tilde{P}_{\tilde{a}_k \tilde{a}_k}^{(\mathbb{T})}).$$

For the last term on the right-hand side of (3.223),

$$\left| \frac{1}{d-1} \sum_{k\in[\![1,\mu]\!]} (\tilde{G}_{1l_k} - \tilde{P}_{1l_k})\tilde{P}_{xl_k}(\tilde{G}^{(\mathbb{T})}_{\tilde{a}_k\tilde{a}_k} - \tilde{P}^{(\mathbb{T})}_{\tilde{a}_k\tilde{a}_k}) \right| \leqslant Cq^{r+2}q^{r+1} \sum_{k\in[\![1,\mu]\!]} |\tilde{P}_{xl_k}| \leqslant C|m_{sc}|q^{r+\ell+3}.$$

where in the first inequality, we used (3.177) for the first factor, and (3.102) for the last factor; in the second inequality, we used (3.164) for the case $x \in \mathbb{T}_1$. For the second term on the right-hand side of (3.223), we have

$$\left| \frac{1}{d-1} \sum_{k\neq m\in[\![1,\mu]\!]} \tilde{G}_{1l_k}\tilde{P}_{xl_m}(\tilde{G}^{(\mathbb{T})}_{\tilde{a}_k\tilde{a}_m} - \tilde{P}^{(\mathbb{T})}_{\tilde{a}_k\tilde{a}_m}) \right| \leqslant Cq^{2\ell+1} \sum_{k\neq m\in[\![1,\mu]\!]} |\tilde{G}^{(\mathbb{T})}_{\tilde{a}_k\tilde{a}_m} - \tilde{P}^{(\mathbb{T})}_{\tilde{a}_k\tilde{a}_m}|$$

$$\leqslant C\omega'^2|m_{sc}|q^{r+2\ell+1} \leqslant C|m_{sc}|q^{r+\ell+1},$$

provided that $\omega'^2 q^\ell \ll 1$, where we used (3.170). Therefore, they together lead to

$$(3.224) \qquad \tilde{G}_{1x} - \tilde{P}_{1x} = \frac{1}{d-1} \sum_{k\in[\![1,\mu]\!]} \tilde{P}_{1l_k}\tilde{P}_{xl_k}(\tilde{G}^{(\mathbb{T})}_{\tilde{a}_k\tilde{a}_k} - \tilde{P}^{(\mathbb{T})}_{\tilde{a}_k\tilde{a}_k}) + O\left(|m_{sc}|q^{r+\ell+1}\right),$$

where the implicit constant depends only on the excess ω and K. Especially, if vertex 1 has radius-R neighborhood, then by Proposition 2.6

$$\tilde{P}_{1x} = -\frac{m_d m_{sc}}{\sqrt{d-1}}, \quad \tilde{P}^{(\mathbb{T})}_{\tilde{a}_k\tilde{a}_k} = m_{sc}, \quad \tilde{P}_{1l_k} = \frac{m_d(-m_{sc})^\ell}{(d-1)^{\ell/2}}, \quad \tilde{P}_{xl_k} = m_d\left(\frac{-m_{sc}}{\sqrt{d-1}}\right)^{\mathrm{dist}_{\tilde{\mathcal{G}}}(x,l_k)}$$

for any index $k \in [\![1, \mu]\!]$. Thus averaging (3.224) over all the vertices x adjacent to 1 (in the following, we write $x \sim 1$ when the vertex x is adjacent to 1), we get

$$
\begin{aligned}
\frac{1}{d} \sum_{x \sim 1} \tilde{G}_{1x} + \frac{m_d m_{sc}}{\sqrt{d-1}} &= \frac{1}{d(d-1)} \sum_{k \in [\![1,\mu]\!]} (\tilde{G}^{(\mathbb{T})}_{\tilde{a}_k \tilde{a}_k} - \tilde{P}^{(\mathbb{T})}_{\tilde{a}_k \tilde{a}_k}) \sum_{x \sim 1} \tilde{P}_{1l_k} \tilde{P}_{xl_k} + O\left(|m_{sc}| q^{r+\ell+1}\right) \\
&= \frac{m_d(-m_{sc})^{\ell}}{d(d-1)^{\ell/2+1}} \sum_{k \in [\![1,\mu]\!]} (\tilde{G}^{(\mathbb{T})}_{\tilde{a}_k \tilde{a}_k} - \tilde{P}^{(\mathbb{T})}_{\tilde{a}_k \tilde{a}_k}) \sum_{x \sim 1} \tilde{P}_{xl_k} + O\left(|m_{sc}| q^{r+\ell+1}\right) \\
&= \frac{-m_d^2 m_{sc}^{2\ell-1}(1 + m_{sc}^2)}{d(d-1)^{\ell+1/2}} \sum_{k \in [\![1,\mu]\!]} (\tilde{G}^{(\mathbb{T})}_{\tilde{a}_k \tilde{a}_k} - \tilde{P}^{(\mathbb{T})}_{\tilde{a}_k \tilde{a}_k}) + O\left(|m_{sc}| q^{r+\ell+1}\right) \\
&= \frac{-m_d^2 m_{sc}^{2\ell-1}(1 + m_{sc}^2)}{\sqrt{d-1}} (Q(\tilde{\mathcal{G}}) - m_{sc}) + O_{\leqslant}\left(\frac{16(\log N)^{1/2+\delta}|m_{sc}| q^{r+1}}{\sqrt{d(d-1)^{\ell}}}\right).
\end{aligned}
$$

In the third line, we used the fact that for any index $k \in [\![1, \mu]\!]$, among the d children of vertex 1, one of them is distance $\ell - 1$ to the vertex l_k, and the others are distance $\ell + 1$ to the vertex l_k. In the last line, we used Proposition (3.40), and $|m_d^2 m_{sc}^{2\ell-1}(1 + m_{sc}^2)| \leqslant 4$. This finishes the proof of Proposition 3.47. □

3.11. **Proof of main results.** In this section, we use the estimates established in the previous sections to prove Theorem 3.1.

3.11.1. *Summary of estimates.* By combining the propositions of the previous sections, we obtain the following sequence of propositions, relating the sets

$$
\Omega^-(z, \ell) \subset \Omega(z, \ell) \subset \Omega_1^+(z, \ell) \subset \bar{\Omega} \subset \mathsf{G}_{N,d}, \quad \Omega_1'(z, \ell) \subset \bar{\Omega} \subset \mathsf{G}_{N,d},
$$

defined in Section 3.1.2. We also recall the parameters from Section 3.1.1, assume that

$$
(3.225) \qquad \qquad \ell \in [\![\ell_*, 2\ell_*]\!], \quad r = 2\ell + 1,
$$

and (3.60), namely that

(3.226)
$$\omega' = \lfloor \log N \rfloor, \quad M = (d-1)^{9\ell}(\log N)^{\delta}.$$

Since, for $|z| \geqslant 2d - 1$, the claim of Theorem 3.1 follows from Proposition 3.3, it suffices to prove the claim of Theorem 3.1 on the following slightly smaller domain

(3.227)
$$\mathcal{D}^* := \left\{ z \in \mathbb{C}_+ : |z| \leqslant 2d, \quad \mathrm{Im}[z] \geqslant \frac{(\log N)^{48\alpha+1}}{N}, \quad |z \pm 2| \geqslant (\log N)^{-\alpha/2+1} \right\},$$

which is the intersection of $\mathcal{D}$ (as in (1.4)) with $\{z \in \mathbb{C}_+ : |z| \leqslant 2d\}$.

Proposition 3.48 (Initial estimates)**.** *Under the assumptions of Theorem 3.1, and the choices of parameters given in (3.225) and (3.226), for $N \geqslant N(\omega, d, \delta)$ large enough, we have*

(3.228)
$$\mathbb{P}(\bar{\Omega}) = 1 - o(N^{-\omega+\delta}).$$

Moreover, for any $z \in \mathbb{C}_+$ such that $|z| \geqslant 2d - 1$, we have $\bar{\Omega} \subset \Omega^-(z, \ell)$.

Proof. The estimate (3.228) follows from Proposition 2.1, and the inclusion $\bar{\Omega} \subset \Omega^-(z, \ell)$ from Proposition 3.3. □

Given a graph $\mathcal{G}$ and a vertex i, we resample the edge boundary of $\mathcal{B}_\ell(i, \mathcal{G})$ using switchings; without loss of generality we assume $i = 1$. Denote the resampled graph by $T_{\mathbf{S}}(\mathcal{G})$ (which depends on the choice of i); $\mathbf{S}$ is the resampling data (whose distribution depends on $\mathcal{G}$).

Proposition 3.49 (Stability under resampling)**.** *Under the assumptions of Theorem 3.1, and the choices of parameters given in (3.225) and (3.226), for $z \in \mathcal{D}^*$, $N \geqslant N(\alpha, \omega, d, \delta)$ large enough, and any $\mathcal{G} \in \Omega(z, \ell)$, the following holds. (i) $\mathcal{G} \in \Omega_1^+(z, \ell)$. (ii) There exists a set $F(\mathcal{G}) \subset \mathsf{S}(\mathcal{G})$ with $\mathbb{P}_{\mathcal{G}}(F(\mathcal{G})) = 1 - o(N^{-\omega+\delta})$ such that for any $\mathbf{S} \in F(\mathcal{G})$ with $T_{\mathbf{S}}(\mathcal{G}) \in \bar{\Omega}$, we have $T_{\mathbf{S}}(\mathcal{G}) \in \Omega_1^+(z, \ell)$.*

Proof. The first statement $\mathcal{G} \in \Omega_1^+(z, \ell)$ follows from Proposition 3.21, and the second statement follows from Proposition 3.26 with $K = 2$. $\qquad \square$

Proposition 3.50 (Improvement under resampling)**.** *Under the assumptions of Theorem 3.1, and the choices of parameters given in (3.225) and (3.226), for $z \in \mathcal{D}^*$, $N \geqslant N(\alpha, \omega, d, \delta)$ large enough, and any $\mathcal{G} \in \Omega_1^+(z, \ell)$, there exists a set $F'(\mathcal{G}) \subset \mathsf{S}(\mathcal{G})$ with $\mathbb{P}_{\mathcal{G}}(F'(\mathcal{G})) = 1 - \mathrm{o}(N^{-\omega+\delta})$ such that for any $\mathbf{S} \in F'(\mathcal{G})$ with $T_{\mathbf{S}}(\mathcal{G}) \in \bar{\Omega}$, we have $T_{\mathbf{S}}(\mathcal{G}) \in \Omega_1'(z, \ell)$.*

Proof. The definition of the set $F'(\mathcal{G})$ and its properties are given in Proposition 3.40. The final statement $T_{\mathbf{S}}(\mathcal{G}) \in \Omega_1'(z, \ell)$ follows from Propositions 3.26 and 3.47 by taking $K = 2^{10}$. $\qquad \square$

The improvement under resampling above applies to the switched graphs $T_{\mathbf{S}}(\mathcal{G})$. However, by general properties of T, it implies an improvement on the original space of graphs.

Proposition 3.51 (Improvement on original space)**.** *Under the assumptions of Theorem 3.1, and the choices of parameters given in (3.225) and (3.226), for $z \in \mathcal{D}^*$, we have*

$$(3.229) \qquad \mathbb{P}(\Omega(z, \ell) \setminus (\Omega(z, \ell) \cap \Omega_1'(z, \ell))) = \mathrm{o}(N^{-\omega+\delta}).$$

Proof. By Propositions 3.48–3.50, the conditions of Proposition 3.9 are satisfied with $q_0, q_1, q_2 = \mathrm{o}(N^{-\omega+\delta})$, and $\bar{\Omega}$ as in Section 3.1, $\Omega = \Omega(z, \ell)$, $\Omega^+ = \Omega_1^+(z, \ell)$, and $\Omega' = \Omega_1'(z, \ell)$. Therefore, Proposition 3.9 implies

$$\mathbb{P}(\Omega(z, \ell) \setminus (\Omega(z, \ell) \cap \Omega_1'(z, \ell))) = \mathrm{o}(N^{-\omega+\delta}),$$

which was the claim. $\qquad \square$

Clearly, by the same argument or by symmetry, (3.229) also holds with vertex 1 replaced by any other vertex $i \in [\![N]\!]$. In particular, for any graph in the intersection of the $\Omega_i'(z, \ell)$ over $i \in [\![N]\!]$, we have the following improved estimates for the entries of its Green's function.

Proposition 3.52 (Self-consistent equation)**.** *Under the assumptions of Theorem 3.1, and the choices of parameters given in (3.225) and (3.226), for any $z \in \mathcal{D}^*$ (as in (3.227)) and $N \geqslant N(\alpha, \omega, d, \delta)$ large enough, we have*

$$(3.230) \quad Q(\mathcal{G}) - m_{sc} = \frac{d-2}{d-1} m_d m_{sc}^{2\ell+1} (Q(\mathcal{G}) - m_{sc}) + O\left(\frac{(\log N)^{1/2+\delta} |m_{sc}| q^r}{(d-1)^{(\ell+1)/2}}\right).$$

Proof. As noted above, the same statement as in Proposition 3.51 holds with vertex 1 replaced by any other vertex $i \in [\![N]\!]$. On the union of the $\Omega_i'(z, \ell)$, the improvement then holds for all i simultaneously, and by a union bound

$$\mathbb{P}(\Omega(z, \ell) \setminus \cap_{i \in [\![N]\!]} \Omega_i'(z, \ell)) \leqslant \sum_{i=1}^{N} \mathbb{P}(\Omega(z, \ell) \setminus \Omega_i'(z, \ell)) = o(N^{-\omega+1+\delta}).$$

For any graph $\mathcal{G} \in \cap_{i \in [\![N]\!]} \Omega_i'(z, \ell)$, by the definition of $\Omega_i'(z, \ell)$ (as in Section 3.1), we have

(3.231)

$$G_{ii}(\mathcal{G}, z) = P_{ii}(\mathcal{E}_r(i, i, \mathcal{G}), z) + \frac{m_d^2 m_{sc}^{2\ell} \mu}{(d-1)^{(\ell+1)}} (Q(\mathcal{G}) - m_{sc}) + O_{\leqslant} \left(2^{2\omega+40} |m_{sc}| q^{r+1}\right),$$

and, for any $j \neq i$, we have the bound

$$(3.232) \qquad |G_{ij}(\mathcal{G}, z) - P_{ij}(\mathcal{E}_r(i, j, \mathcal{G}), z)| \leqslant (\omega+1) 2^{2\omega+44} |m_{sc}| q^{r+1}.$$

In the following, we derive an approximate self-consistent equation for $Q(\mathcal{G}) - m_{sc}$. Using the Green's function identity (B.5), notice that

$$Q(\mathcal{G}) - m_{sc} = \frac{1}{Nd} \sum_{(i,j) \in \vec{E}} (G_{jj}^{(i)} - m_{sc}) = \frac{1}{Nd} \sum_{(i,j) \in \vec{E}} \left(G_{jj} - \frac{G_{ij}G_{ij}}{G_{ii}} - m_{sc} \right)$$

$$(3.233) \qquad\qquad = \frac{1}{N} \sum_i \left(G_{ii} - \frac{1}{d} \sum_{j:j\sim i} \frac{G_{ij}G_{ij}}{G_{ii}} - m_{sc} \right),$$

where $\vec{E}$ is the set of oriented edges of $\mathcal{G}$, and where here $j \sim i$ means that the vertices i and j are adjacent to each other. Since $\mathcal{G} \in \bar{\Omega}$, at least $N - N^\delta$ of the vertices of $\mathcal{G}$ have radius-R tree neighborhoods. The contribution to $Q(\mathcal{G}) - m_{sc}$ from those vertices which do not have radius-R tree neighborhoods is $O(N^{\delta-1})$. For any vertex i that has radius-R tree neighborhood, by the definition of $\Omega_i'(z, \ell)$,

(3.234)
$$G_{ii} - m_d = m_d^2 m_{sc}^{2\ell} \frac{d}{d-1}(Q(\mathcal{G}) - m_{sc}) + O\left(\frac{(\log N)^{1/2+\delta}|m_{sc}|q^r}{(d-1)^{(\ell+1)/2}} \right),$$

(3.235)
$$\frac{1}{d} \sum_{j\sim i} G_{ij} + \frac{m_d m_{sc}}{\sqrt{d-1}} = \frac{-m_d^2 m_{sc}^{2\ell-1}(1 + m_{sc}^2)}{\sqrt{d-1}}(Q(\mathcal{G}) - m_{sc}) + O\left(\frac{(\log N)^{1/2+\delta}|m_{sc}|q^{r+1}}{(d-1)^{(\ell+1)/2}} \right).$$

Also, by the stability estimate Claim 3.39, for any vertex i with radius-R tree neighborhood, and vertex j adjacent to i, we have $|G_{ii} - m_d| = O(|m_{sc}|q^r)$ and $|G_{ij} - m_d m_{sc}/\sqrt{d-1}| = O(|m_{sc}|q^r)$, where the implicit constant depends only on ω. It follows that

(3.236)
$$\frac{G_{ij}G_{ij}}{G_{ii}} - \frac{m_d m_{sc}^2}{d-1} = \frac{-\frac{2m_d^2 m_{sc}}{\sqrt{d-1}}(G_{ij} + \frac{m_d m_{sc}}{\sqrt{d-1}}) - \frac{m_d^2 m_{sc}^2}{d-1}(G_{ii} - m_d) + m_d(G_{ij} + \frac{m_d m_{sc}}{\sqrt{d-1}})^2}{m_d G_{ii}}$$

$$= -\frac{2m_{sc}}{\sqrt{d-1}}(G_{ij} + \frac{m_d m_{sc}}{d-1}) - \frac{m_{sc}^2}{d-1}(G_{ii} - m_d) + O(q^{2r}).$$

Combining (3.234)–(3.236), we get

$$G_{ii} - \frac{1}{d} \sum_{j:j\sim i} \frac{G_{ij}G_{ij}}{G_{ii}} - m_{sc} = \frac{d-2}{d-1} m_{sc}^{2\ell+1} m_d(Q(\mathcal{G}) - m_{sc}) + O\left(\frac{(\log N)^{1/2+\delta}|m_{sc}|q^r}{(d-1)^{(\ell+1)/2}}\right),$$

for all vertices i which have radius-R tree neighborhoods. Then averaging over $i \in [\![N]\!]$, we obtain (3.230), as claimed. $\qquad\Box$

The equation (3.230) implies

$$(3.237) \qquad Q(\mathcal{G}) - m_{sc} = \left(1 - \frac{d-2}{d-1} m_d m_{sc}^{2\ell+1}\right)^{-1} O\left(\frac{(\log N)^{1/2+\delta}|m_{sc}|q^r}{(d-1)^{(\ell+1)/2}}\right),$$

provided that the term in the first bracket does not vanish. To use this equation to show that the left-hand side is small, we require a lower bound on the term in the first bracket on the right-hand side. Since $1 - (d-2)m_d m_{sc}^{2\ell+1}/(d-1)$ may be zero on the spectral domain $\mathcal{D}^*$, such a bound only holds on an ℓ-dependent subset of the spectral domain, which we now define. (In Section 3.11.2, we will use the flexibility in the choice of $\ell \in [\![\ell_*, 2\ell_*]\!]$ to recover the entire spectral domain.)

First, we define the Joukowsky transform ϕ to be the holomorphic bijection from the upper half unit disk $\mathbb{D}_+$ to the upper half plane $\mathbb{C}_+$ given by

$$\phi : w \in \mathbb{D}_+ \mapsto -\left(w + w^{-1}\right) \in \mathbb{C}_+.$$

It is the functional inverse of $z \mapsto m_{sc}(z)$, i.e. $m_{sc}(\phi(w)) = w$. For any $\ell \in [\![\ell_*, 2\ell_*]\!]$ as in (3.5), we define the small error parameter

$$(3.238) \qquad\qquad \varepsilon_\ell := \frac{(\log N)^{1/2+2\delta}}{(d-1)^{(\ell+1)/2}} \ll (\log N)^{1-\alpha/2},$$

as well as the sets $\tilde{\Lambda}_\ell \subset \mathbb{D}_+$ and $\Lambda_\ell \subset \mathbb{C}_+$ by

(3.239)

$$\tilde{\Lambda}_\ell := \left\{ m_{sc}(z) : z \in \mathbb{C}_+, \left|1 - \frac{d-2}{d-1} m_{sc}^{2\ell+1}(z) m_d(z)\right| \geq \varepsilon_\ell \right\}, \qquad \Lambda_\ell := \phi(\tilde{\Lambda}_\ell).$$

Proposition 3.53 (Self-consistent equation). *Under the assumptions of Theorem 3.1, and the choices of parameters given in (3.225) and (3.226), for any $z \in \mathcal{D}^*$ (as in (3.227)) and $N \geqslant N(\alpha, \omega, d, \delta)$ large enough, we have*

$$\mathbb{P}(\Omega(z, \ell) \setminus \cap_{i \in [\![N]\!]} \Omega_i'(z, \ell)) = \mathrm{o}(N^{-\omega+1+\delta}).$$

Moreover, for $z \in \mathcal{D}^* \cap \Lambda_\ell$ and any $\mathcal{G} \in \bigcap_{i \in [\![N]\!]} \Omega_i'(z, \ell)$, the normalized Green's function of $\mathcal{G}$ satisfies, for any $i, j \in [\![N]\!]$,

$$|G_{ij}(\mathcal{G}, z) - P_{ij}(\mathcal{E}_r(i, j, \mathcal{G}), z)| \leqslant (\omega + 1)2^{2\omega+44}|m_{sc}|q^{r+1},$$

where $r = 2\ell + 1$.

Proof. Let $z \in \mathcal{D}^* \cap \Lambda_\ell$. Then by the definition of the set $\tilde{\Lambda}_\ell$ in (3.239), we have

$$\left| 1 - \frac{d-2}{d-1} m_{sc}^{2\ell+1} m_d \right| \geqslant \frac{(\log N)^{1/2+2\delta}}{(d-1)^{(\ell+1)/2}},$$

and (3.237) implies

$$|Q(\mathcal{G}) - m_{sc}| = O\left(|m_{sc}|q^r (\log N)^{-\delta} \right),$$

where the implicit constant depends only on ω. Plugging the above expression into (3.231), we get

$$G_{ii}(\mathcal{G}, z) = P_{ii}(\mathcal{E}_r(i, i, \mathcal{G}), z) + \mathrm{O}_\leqslant \left((1 + 2^{2\omega+40})|m_{sc}|q^{r+1} \right),$$

for N large enough. This finishes the proof of Proposition 3.53 by combining with (3.232). □

3.11.2. *Decomposition of the spectral domain.* The following lemma gives a precise description of the sets $\tilde{\Lambda}_\ell$, stating that, except for two small regions near ± 1, the half disk $\mathbb{D}_+$ is contained in the union of the sets $\tilde{\Lambda}_\ell$, i.e., in $\cup_{\ell \in [\![\ell_*, 2\ell_*]\!]} \tilde{\Lambda}_\ell$. To be precise,

we define the spectral domains

$$(3.240) \quad \mathcal{D}_\ell = \{z \in \mathbb{C}_+ : |z| \leqslant 2d, \quad \mathrm{Im}[z] \geqslant (d-1)^{24\ell} \log N/N, \quad |z \pm 2| \geqslant 4\varepsilon_\ell\},$$

$$(3.241) \quad \tilde{\mathcal{D}}_\ell = \mathbb{D}_+ \setminus \left\{ w = e^{i\theta} r \in \mathbb{D}_+ : |\theta| \leqslant \varepsilon_\ell, 1 - \varepsilon_\ell \leqslant r < 1 \right\}.$$

Lemma 3.54. *For any $\ell \in [\![\ell_*, 2\ell_*]\!]$, define $\tilde{\Lambda}_\ell$ as in (3.239). Then $\mathbb{D}_+ \setminus \tilde{\Lambda}_\ell$ is contained in*

$$\left\{ w = e^{i\theta} r : 0 < \theta \leqslant \varepsilon_\ell, 1 - \varepsilon_\ell \leqslant r < 1 \right\} \cup \left\{ w = e^{\pi i} e^{i\theta} r : -\varepsilon_\ell \leqslant \theta < 0, 1 - \varepsilon_\ell \leqslant r < 1 \right\} \cup$$

$$\bigcup_{k=1}^{\ell} \left\{ w = e^{\frac{k\pi i}{\ell+1}} e^{i\theta} r : |\theta| \leqslant \frac{2\pi}{d(\ell+1)}, 0 < r < 1 \right\}.$$

As a consequence, for any $\theta_0 \in (0, \pi)$, there exists some $\ell \in [\![\ell_*, 2\ell_*]\!]$ such that

$$(3.242) \qquad \tilde{\mathcal{D}}_\ell \cap \{w = e^{i\theta_0} r : 0 < r < 1\} \subset \tilde{\Lambda}_\ell.$$

Proof. By the definition of the set $\tilde{\Lambda}_\ell$, its complement is

$$\left| 1 - \frac{d-2}{d-1} m_{sc}^{2\ell+1} \frac{m_{sc}}{1 - m_{sc}^2/(d-1)} \right| < \varepsilon_\ell.$$

This implies that

$$\left| 1 - \frac{m_{sc}^2}{d-1} - \frac{d-2}{d-1} m_{sc}^{2\ell+2} \right| < 2\varepsilon_\ell,$$

and therefore

$$(3.243) \qquad \left| 1 - m_{sc}^{2\ell+2} \right| < \left| \frac{m_{sc}^2}{d-1} - \frac{m_{sc}^{2\ell+2}}{d-1} \right| + 2\varepsilon_\ell < \frac{2}{d-1} + 2\varepsilon_\ell.$$

From direct computation, for any $0 \leqslant r \leqslant 1$ and $\theta \in [-\pi, \pi]$, we have the following simple estimate:

$$(3.244) \qquad \frac{1-r}{2} + \frac{\sqrt{r}|\theta|}{\pi} \leqslant |1 - e^{i\theta} r| \leqslant (1-r) + |\theta|.$$

Therefore (3.243) implies

$$m_{sc} \in \bigcup_{k=0}^{\ell+1} \left\{ w = e^{\frac{k\pi i}{\ell+1}} e^{i\theta} r \in \mathbb{D}_+ : |\theta| \leqslant \frac{2\pi}{d(\ell+1)}, 0 < 1 - r^{2(\ell+1)} \leqslant \frac{5}{d} \right\}.$$

We have better estimates if $m_{sc} = e^{i\theta}r$ or $m_{sc} = e^{i\pi}e^{i\theta}r$, for some $|\theta| \leqslant \frac{2\pi}{d(\ell+1)}$ and $0 < 1 - r^{2(\ell+1)} \leqslant \frac{5}{d}$. In this case, on the complement of $\tilde{\Lambda}_\ell$,

$$\left| \frac{d-2}{d-1}(1 - m_{sc}^{2\ell+2}) \right| \leqslant \left| \frac{1 - m_{sc}^2}{d-1} \right| + 2\varepsilon_\ell.$$

Combining the above expression with (3.244), we get

$$\frac{d-2}{d-1} \left(\frac{1 - r^{2(\ell+1)}}{2} + \frac{2(\ell+1)r^{(\ell+1)}|\theta|}{\pi} \right) \leqslant \frac{1}{d-1}\left(1 - r^2 + 2|\theta| \right) + 2\varepsilon_\ell.$$

It follows that $|\theta| \leqslant \varepsilon_\ell$ and $1 - \varepsilon_\ell \leqslant r < 1$. This finishes the proof of the first statement.

For the second statement, if $\theta_0 \in (0, (1-\frac{2}{d})\frac{\pi}{\ell_*+1}) \cup ((1+\frac{2}{d})\frac{\ell_*\pi}{\ell_*+1}, \pi)$, then $\{z = e^{i\theta_0}r : 0 \leqslant r \leqslant 1\} \cap \tilde{\mathcal{D}}_{\ell_*} \subset \tilde{\Lambda}_{\ell_*}$. In the following we consider the case, $\theta_0 \in [(1 - \frac{2}{d})\frac{\pi}{\ell_*+1}, (1+\frac{2}{d})\frac{\ell_*\pi}{\ell_*+1}]$. We use the convention that $(a \bmod \pi) \in [-\pi/2, \pi/2)$, for any $a \in \mathbb{R}$. If we can find some $\ell \in [\![\ell_*, 2\ell_*]\!]$, such that $((\ell+1)\theta_0 \bmod \pi) \in [-\pi/2, -\pi/8] \cup [\pi/8, \pi/2)$, then there exists some integer k such that

$$-\frac{3\pi}{8(\ell+1)} \leqslant \left| \theta_0 - \frac{k\pi}{\ell+1} - \frac{\pi}{2(\ell+1)} \right| \leqslant \frac{3\pi}{8(\ell+1)},$$

and thus $\{z = e^{i\theta_0}r : 0 < r < 1\} \in \tilde{\Lambda}_\ell$. In the following we prove such ℓ exists. By symmetry we assume $\theta_0 \in [(1 - \frac{2}{d})\frac{\pi}{\ell_*+1}, \frac{\pi}{2}]$. We consider the following numbers,

$$(3.245) \qquad (\ell_* + 1)\theta_0 \bmod \pi, (\ell_* + 2)\theta_0 \bmod \pi, \ldots, (2\ell_* + 1)\theta_0 \bmod \pi.$$

If $((\ell_*+1)\theta_0 \bmod \pi) \in [-\pi/2, -\pi/8] \cup [\pi/8, \pi/2)$, then we can take $\ell = \ell_*$. Otherwise, we assume $((\ell_* + 1)\theta_0 \bmod \pi) \in (-\pi/8, \pi/8)$. Since $(2\ell_* + 1)\theta_0 - (\ell_* + 1)\theta_0 = \ell_*\theta_0 \geqslant (1 - \frac{2}{d})\frac{\ell_*\pi}{\ell_*+1} \geqslant \frac{\pi}{2}$, the above sequence (3.245) can not all stay in the interval

$(-\pi/8, \pi/8)$. Say $(\ell+1)\theta_0 \bmod \pi$ is the first number in the above sequence which is not in $(-\pi/8, \pi/8)$. We can take this ℓ, then $(\ell\theta_0 \bmod \pi) \in (-\pi/8, \pi/8)$, and $((\ell+1)\theta_0 \bmod \pi) \in [-\pi/2, -3\pi/8) \cup [\pi/8, \pi/2)$. This finishes the proof. $\qquad\square$

Lemma 3.55. *(i) For the choice of parameters in (3.225)–(3.226), for any $z \in \mathcal{D}_\ell$, all of the conditions in Propositions 3.3, 3.21, 3.26 and 3.47 are satisfied for $K \in \{2, 2^{10}\}$; i.e.,*

$$\sqrt{d-1} \geqslant \max\{(\omega+1)^2 2^{2\omega+10}, 2^8(\omega+1)K\}, \quad \omega'^2 q^\ell \ll 1, \quad \sqrt{N\mathrm{Im}[z]}q^{3r+2} \gg M.$$

(ii) For any $\ell \in [\![\ell_*, 2\ell_*]\!]$, we have

(3.246) $$\mathcal{D}^* \subset \mathcal{D}_\ell \subset \phi(\tilde{\mathcal{D}}_\ell).$$

Proof. It is straightforward to check that (i) holds. In the following, we therefore only prove (ii). For this, notice that $(d-1)^{24\ell} \leqslant (d-1)^{48\ell_*} \leqslant (\log N)^{48\alpha}$, and that in combination with (3.238), it follows that $\mathcal{D}^* \subset \mathcal{D}_\ell$. For the second inclusion in (3.246), observe that, for any $w = e^{i\theta}r$ such that $0 < \theta \leqslant \varepsilon_\ell$ and $1 - \varepsilon_\ell \leqslant r \leqslant 1$, we have

$$|\phi(w) + 2| = \left| e^{i\theta}r + \frac{1}{e^{i\theta}r} - 2 \right| < 4\varepsilon_\ell,$$

and that we have similar estimates for $w = e^{i\pi}e^{i\theta}r$ with $-\varepsilon_\ell \leqslant \theta \leqslant 0$ and $1 - \varepsilon_\ell \leqslant r \leqslant 1$. Therefore,

$$\{z \in \mathbb{C}_+ : |z \pm 2| \geqslant 4\varepsilon_\ell\} \subset \phi(\tilde{\mathcal{D}}_\ell),$$

and $\mathcal{D}_\ell \subset \phi(\tilde{\mathcal{D}}_\ell)$ follows. This finishes the proof of (3.246). $\qquad\square$

3.11.3. *Proof of Theorem 3.1.* We define a lattice on $\mathbb{D}_+$ by

$$\tilde{L} := \left\{ e^{i\theta}r \in \mathbb{D}_+ : \theta \in \frac{\pi\mathbb{Z}}{N^3}, r \in \frac{\mathbb{Z}}{N^3} \right\}.$$

The image of $\tilde{L}$ under the Joukowsky transform defines a discrete approximation of $\mathcal{D}^*$ by

(3.247)
$$L := \phi(\tilde{L}) \cap \mathcal{D}^*.$$

Notice that $\mathcal{D}^*$ can indeed be well approximated by L, in the sense that for any $z \in \mathcal{D}^*$ there is some $z' \in L$ such that $|z - z'| = (\log N)^{\mathrm{O}(1)}/N^3$. Therefore, by the following claim, we only need to prove Theorem 3.1 for $z \in L$. The claim is a consequence of the Lipschitz property of Green's function.

Claim 3.56. *For any $\ell \in [\![\ell_*, 2\ell_*]\!]$, and $z, z' \in \mathcal{D}^*$ with $|z - z'| = (\log N)^{\mathrm{O}(1)}/N^3$, we have*

$$\Omega^-(z, \ell) \subset \Omega(z', \ell).$$

Proof. For any graph $\mathcal{G} \in \Omega^-(z, \ell)$, the Green's function of its normalized adjacency matrix satisfies

$$|G_{ij}(z) - G_{ij}(z')| \leqslant |z - z'| \sum_{m=1}^{N} |G_{im}(z)G_{mj}(z')| \leqslant (\log N)^{\mathrm{O}(1)}/N,$$

where we used $|z - z'| = (\log N)^{\mathrm{O}(1)}/N^3$, $|G_{im}(z)| = \mathrm{O}(1)$ from the definition (3.7) of $\Omega^-(z, \ell)$ and (2.15), as well as the trivial bound $|G_{mj}(z')| \leqslant 1/\eta \leqslant N$. Moreover, the same estimate holds for $|P_{ij}(\mathcal{E}_r(i, j, \mathcal{G}), z) - P_{ij}(\mathcal{E}_r(i, j, \mathcal{G}), z')|$. As a result,

$$|G_{ij}(z') - P_{ij}(\mathcal{E}_r(i, j, \mathcal{G}), z')| = |G_{ij}(z) - P_{ij}(\mathcal{E}_r(i, j, \mathcal{G}), z)| + (\log N)^{\mathrm{O}(1)}/N \leqslant |m_{sc}|q^r,$$

and the claim follows. $\qquad\qquad\qquad\square$

Claim 3.57. *For any $\ell \in [\![\ell_*, 2\ell_*]\!]$, we have*

$$\Omega^-(z, \ell) \subset \Omega^-(z, \ell_*),$$

provided that $\sqrt{d - 1} \geqslant 2^{2\omega+3}$.

Proof. Let $\mathcal{G} \in \Omega^-(z, \ell)$. Then, by (2.14) in Proposition 2.7,

$$|G_{ij}(z) - P_{ij}(\mathcal{E}_{r_*}(i, j, \mathcal{G}), z)|$$

$$\leqslant |G_{ij}(z) - P_{ij}(\mathcal{E}_r(i, j, \mathcal{G}), z)| + |P_{ij}(\mathcal{E}_{r_*}(i, j, \mathcal{G}), z) - P_{ij}(\mathcal{E}_r(i, j, \mathcal{G}), z)|$$

$$\leqslant \frac{1}{2}|m_{sc}|q^r + \delta_{r \neq r_*} 2^{2\omega+3}|m_{sc}|q^{2\ell_*+2} \leqslant \frac{1}{2}|m_{sc}|q^{r_*},$$

provided that $\sqrt{d-1} \geqslant 2^{2\omega+3}$. Therefore, $\mathcal{G} \in \Omega^-(z, \ell_*)$, as claimed. $\qquad\square$

Proof of Theorem 3.1. For any $\theta_0 \in \frac{\pi \mathbb{Z}}{N^3} \cap (0, \pi)$, the Joukowsky transform ϕ sends the ray $\{w = e^{i\theta_0}r : 0 < r < 1\}$ to a branch of some hyperbola. With θ_0 fixed, we consider the set

$$\left\{ r \in \frac{\mathbb{Z}}{N^3} : \phi(e^{i\theta_0}r) \in \mathcal{D}^* \right\} = \left\{ \frac{k_0}{N^3}, \frac{k_0+1}{N^3}, \frac{k_0+2}{N^3}, \dots, \frac{k_1}{N^3} \right\},$$

for some $0 < k_0 \leqslant k_1 < N^3$, and denote

$$z_k = \phi\left(\frac{e^{i\theta_0}k}{N^3} \right), \quad 1 \leqslant k \leqslant N^3.$$

One can check that $k_0/N^3 \geqslant 1/(3d)$, $|z_{k_0}| \geqslant 2d-1$, and $|z_{k+1} - z_k| \leqslant 10d^2/N^3$ for $k_0 \leqslant k \leqslant k_1$.

By Proposition 3.54, there exists some $\ell \in [\![\ell_*, 2\ell_*]\!]$ such that $\tilde{\mathcal{D}}_\ell \cap \{e^{i\theta_0}r : 0 < r < 1\} \subset \tilde{\Lambda}_\ell$. Therefore, combining with (3.246), we know that $z_{k_0}, z_{k_0+1}, \dots, z_{k_1} \in \Lambda_\ell$. By Proposition 3.48, $\bar{\Omega} \subset \Omega^-(z_{k_0}, \ell)$, and

$$(3.248) \qquad\qquad \mathbb{P}(\Omega^-(z_{k_0}, \ell)) = 1 - o(N^{-\omega+\delta}).$$

For any $k_0 \leqslant k \leqslant k_1 - 1$, it follows from Claim 3.56 that

$$(3.249) \qquad\qquad \Omega^-(z_k, \ell) \subset \Omega(z_{k+1}, \ell).$$

By Proposition 3.53, we have

(3.250)
$$\mathbb{P}(\Omega(z_{k+1}, \ell) \setminus \cap_{i \in [\![N]\!]} \Omega_i'(z_{k+1}, \ell)) = o(N^{-\omega+1+\delta}),$$

and

(3.251)
$$\cap_{i \in [\![N]\!]} \Omega_i'(z_{k+1}, \ell) \subset \Omega^-(z_{k+1}, \ell),$$

provided that $\sqrt{d-1} \geqslant (\omega+1)2^{2\omega+45}$. It follows from combining (3.249)–(3.251) that

(3.252)
$$\mathbb{P}(\Omega^-(z_k, \ell) \setminus \Omega^-(z_{k+1}, \ell)) = o(N^{-\omega+1+\delta}).$$

By definition, on the set $\cap_{k=k_0}^{k_1} \Omega^-(z_k, \ell)$, we have

$$|G_{ij}(z) - P_{ij}(\mathcal{E}_r(i, j, \mathcal{G}), z)| \leqslant \frac{1}{2}|m_{sc}|q^r,$$

for any $z = z_{k_0}, z_{k_0+1}, \ldots, z_{k_1}$. Moreover, combining (3.248) and (3.252), the above holds with high probability,

(3.253)
$$\mathbb{P}(\cap_{k=k_0}^{k_1} \Omega^-(z_k, \ell)) = 1 - (k_1 - k_0 + 1) o(N^{-\omega+1+\delta}) = 1 - o(N^{-\omega+4+\delta}).$$

Combining with Claim 3.57, the estimate (3.253) implies

$$\mathbb{P}(\cap_{k=k_0}^{k_1} \Omega^-(z_k, \ell_*)) = 1 - o(N^{-\omega+4+\delta}).$$

The above argument is independent of $\theta_0 \in \frac{\mathbb{Z}}{N^3} \cap (0, \pi)$. Thus, by a union bound, with probability at least $1 - o(N^{-\omega+7+\delta})$, uniformly in $z \in L$, we have

(3.254)
$$|G_{ij}(z) - P_{ij}(\mathcal{E}_{r_*}(i, j, \mathcal{G}), z)| \leqslant \frac{1}{2}|m_{sc}|q^{r_*}.$$

Since for any $z \in \mathcal{D}^*$, there is some $z' \in L$ such that $|z - z'| = (\log N)^{O(1)}/N^3$, the Lipschitz property of Green's function, Claim 3.56, implies that the above estimate

(3.254) holds uniformly for $z \in \mathcal{D}^*$, with possibly a slightly larger constant:

$$(3.255) \qquad |G_{ij}(z) - P_{ij}(\mathcal{E}_{r_*}(i, j, \mathcal{G}), z)| \leqslant |m_{sc}| q^{r_*}.$$

This is (3.2), and thus the proof of Theorem 3.1 is complete. $\qquad \square$

4. Bulk Universality

4.1. **Strategy of proof.** Our goal is to prove that, on the regime $1 \ll d \ll N^{2/3}$, in the bulk on the spectrum, the local eigenvalue statistics of $A/\sqrt{d-1}$ are the same as those of the GOE. As mentioned in Section 1.2, in order to show this, we interpolate between the RRG and the GOE using Dyson Brownian motion, or more precisely its Ornstein-Uhlenbeck version.

4.1.1. *Constrained Dyson Brownian motion.* The adjacency matrix A of a regular graph is subject to the hard constraints that its rows and columns have fixed sum (i.e. it has the eigenvector $\mathbf{e} = N^{-1/2}(1, \ldots, 1)^*$). Therefore, instead of the usual Dyson Brownian motion, we use Dyson Brownian motion constrained to the subspace of symmetric matrices whose row and column sums vanish.

We begin with the notion of an Ornstein-Uhlenbeck process on a general finite-dimensional space.

Definition 4.1. *Let $\mathcal{H}$ be a real finite-dimensional Hilbert space. Let $(\mathbf{f}_\alpha)_\alpha$ be an orthonormal basis of $\mathcal{H}$.*

(i) Let $(w_\alpha)_\alpha$ be i.i.d. standard normal random variables. Then we define the standard Gaussian measure on $\mathcal{H}$ as $W := \sum_\alpha w_\alpha \mathbf{f}_\alpha$.

(ii) Let $(h_\alpha)_\alpha$ be i.i.d. Ornstein-Uhlenbeck processes satisfying

$$\mathrm{d}h_\alpha = \mathrm{d}B_\alpha - \frac{1}{2} h_\alpha \, \mathrm{d}t \,,$$

where $(B_\alpha)_\alpha$ is a family of i.i.d. standard Brownian motions. Then we define the standard Ornstein-Uhlenbeck process on $\mathcal{H}$ *as* $H(t) := \sum_\alpha h_\alpha(t)\,\mathbf{f}_\alpha.$

It is easy to verify that the laws of W and the process H do not depend on the choice of the orthonormal basis $(\mathbf{f}_\alpha)$, and that the standard Gaussian measure is invariant under the standard Ornstein-Uhlenbeck process. We use these properties tacitly from now on.

For example, let $\mathcal{H} := \{H \in \mathbb{R}^{N \times N} : H = H^*\}$ be the Hilbert space of real symmetric $N \times N$ matrices with inner product

$$(4.1) \qquad\qquad \langle X, Y \rangle := \frac{N}{2} \operatorname{Tr}(XY).$$

Then the usual N-dimensional Dyson Brownian motion is the standard Ornstein-Uhlenbeck process $H(t)$ on $\mathcal{H}$. More explicitly, $H(t)$ is the Markov process satisfying the SDE

$$(4.2) \qquad\qquad \mathrm{d}H \;=\; \frac{1}{\sqrt{N}}\,\mathrm{d}B - \frac{1}{2}H\,\mathrm{d}t,$$

where $B(t)$ is Brownian motion on the space of $N \times N$ real symmetric matrices with quadratic covariation $\langle B_{ij}, B_{kl} \rangle(t) = (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})t$.

More intrinsically, given a finite-dimensional Hilbert space V, we denote the Hilbert space of symmetric linear maps on V with inner product (4.1) by $\mathcal{H}(V)$. Then we define *Dyson Brownian motion (DBM) on V* to be the standard Ornstein-Uhlenbeck process on $\mathcal{H}(V)$. With this point of view, the usual N-dimensional DBM is the DBM on $V = \mathbb{R}^N$, and the constrained DBM is the DBM on $V = \mathbf{e}^\perp$. Note that the normalization N in (4.1) does not need to agree with the dimension of V, which is $N - 1$ for $V = \mathbf{e}^\perp$. We make the convention to always normalize the inner product (4.1) by N, no matter the dimension of V, and always denote the dimension of V by M. Finally, we denote the inner product on V by $\mathbf{v} \cdot \mathbf{w}$ for $\mathbf{v}, \mathbf{w} \in V$.

Definition 4.2 (Constrained DBM and GOE). *The* constrained DBM *is the DBM on* $\mathbf{e}^\perp$, *i.e. the standard Ornstein-Uhlenbeck process on* $\mathcal{H}(\mathbf{e}^\perp)$ *with inner product* (4.1). *The* constrained GOE *is the standard Gaussian measure on* $\mathcal{H}(\mathbf{e}^\perp)$ *with inner product* (4.1).

Thus, up to a change of basis, the constrained DBM is equivalent to the usual $(N-1)$-dimensional DBM, with the minor difference of normalization by N rather than $N-1$. However, since the definition of the d-regular graph is tied to the standard basis of $\mathbb{R}^N$, it is frequently convenient to work with the constrained DBM in the standard basis of $\mathbb{R}^N$.

Next, in accordance with the decomposition $\mathbb{R}^N = \mathbf{e}^\perp \oplus \operatorname{span}(\mathbf{e})$, we have a canonical isomorphism $H \mapsto \tilde{H} := H \oplus 0$ from $\mathcal{H}(\mathbf{e}^\perp)$ to the set of matrices

$$(4.3) \qquad\qquad \mathcal{M} \; := \; \{H \in \mathbb{R}^{N \times N} : H = H^*, H\mathbf{e} = 0\}.$$

Throughout this paper, we tacitly identify H and $\tilde{H}$.

We denote by $\mathcal{C}^n(\mathcal{M})$ the space of functions $F : \mathcal{M} \to \mathbb{C}$ with continuous bounded derivatives up to order n. Sometimes it will be convenient to compute derivatives of functions $F \in \mathcal{C}^n(\mathcal{M})$ in directions of $\mathbb{R}^{N \times N}$ that do not lie in $\mathcal{M}$, which is made possible by the following convention.

Definition 4.3. *Let* $P = I - \mathbf{e}\mathbf{e}^*$ *be the orthogonal projection from* $\mathbb{R}^N$ *onto* $\mathbf{e}^\perp$. *We extend any function* $F \in \mathcal{C}^n(\mathcal{M})$ *to a* $\mathcal{C}^n$*-function on* $\mathbb{R}^{N \times N}$ *through*

$$H \; \longmapsto \; F\left(\frac{1}{2}P(H + H^*)P\right),$$

and denote this extended function also by F. *Finally, for any* $F \in \mathcal{C}^1(\mathcal{M})$ *and* $i, j \in [\![1, N]\!]$, *we use the abbreviation* $\partial_{ij}F(H) \equiv \frac{\partial F}{\partial H_{ij}}(H)$.
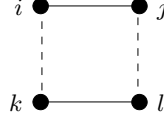
FIGURE 12. A simple switching is given by replacing the solid edges by the dashed edges.

From now on, we take W to be the constrained GOE and $H \equiv H(t)$ to be the constrained DBM, with initial condition

$$(4.4) \qquad H(0) := \frac{1}{\sqrt{d-1}}(A - d\mathbf{e}\mathbf{e}^*) \in \mathcal{M}.$$

Here A is the adjacency matrix of the random d-regular graph. In particular, the eigenvalues of $H(0)$ as an element of $\mathcal{H}(\mathbf{e}^\perp)$ are the rescaled nontrivial eigenvalues of A.

4.1.2. *Switchings.* Simple switchings are an especially convenient generating set of $\mathcal{M}$; they play a central role throughout this paper. For any $i, j, k, l \in [\![1, N]\!]$ we define the *simple switching* $\xi_{ij}^{kl} \in \mathcal{M}$ by

$$(4.5) \qquad \xi_{ij}^{kl} := \Delta_{ij} + \Delta_{kl} - \Delta_{ik} - \Delta_{jl} \qquad \text{where} \quad (\Delta_{ij})_{pq} := \delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp} \, .$$

The action of a simple switching ξ_{ij}^{kl} on an adjacency matrix, given by $A \mapsto A + \xi_{ij}^{kl}$, amounts to adding the edges $\{i, j\}, \{k, l\}$ and removing the edges $\{i, k\}, \{j, l\}$; this is illustrated in Figure 12 and made precise in (4.29) below. In this section, the four vertices need not be distinct.

Next, we define the abbreviations

$$(4.6) \qquad H_{ij}^{kl} := \mathrm{Tr}(\xi_{ij}^{kl} H) \, , \qquad \partial_{ij}^{kl} := \partial_{\xi_{ij}^{kl}} = \mathrm{Tr}(\xi_{ij}^{kl} \partial) \, ,$$

for all $i, j, k, l \in [\![1, N]\!]$. Here ∂_X denotes the directional derivative in the direction X. Explicitly, expressed in the standard basis on $\mathbb{R}^N$, we have

(4.7) $$H_{ij}^{kl} = 2(H_{ij} + H_{kl} - H_{ik} - H_{jl}),$$

(4.8) $$\partial_{ij}^{kl} F(H) = 2(\partial_{ij} + \partial_{kl} - \partial_{ik} - \partial_{jl}) F(H),$$

where $F \in \mathcal{C}^1(\mathcal{M})$. With these abbreviations, the generator of the constrained DBM can be expressed in terms of switchings as stated in the following proposition.

Proposition 4.4. *The generator of the constrained DBM from Definition 4.2 is*

(4.9) $$L := \frac{1}{16 N^3} \sum_{i,j,k,l} (\partial_{ij}^{kl})^2 - \frac{1}{32 N^2} \sum_{i,j,k,l} H_{ij}^{kl} \partial_{ij}^{kl}.$$

This means that for any $F \in \mathcal{C}^2(\mathcal{M})$ we have

(4.10) $$\frac{\mathrm{d}}{\mathrm{d}t} \mathbb{E}[F(H(t))] = \mathbb{E}[LF(H(t))].$$

Proof. Let $\hat{H}(t)$ be the standard Ornstein-Uhlenbeck process from Definition 4.1 on the space $\mathcal{H}(\mathbb{R}^{N-1})$ with inner product (4.1). As in the example (4.2), we obtain the quadratic covariation

(4.11) $$\langle \hat{H}_{ij}, \hat{H}_{kl} \rangle(t) = \frac{1}{N}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})t.$$

Next, let $R \in \mathrm{O}(N)$ satisfy $R\mathbf{e}_N = \mathbf{e}$. Then, since the inner product (4.1) is invariant under orthogonal conjugations, we can express the constrained DBM as $H(t) = R(\hat{H}(t) \oplus 0)R^*$. We abbreviate $H \equiv H(t)$ and write for $F \in \mathcal{C}^2(\mathcal{M})$, using Itô calculus,

$$\mathrm{d}\,\mathbb{E}F(H) = -\frac{1}{2}\sum_{i,j} \mathbb{E}\big[H_{ij}(\partial_{ij}F)(H)\big]\,\mathrm{d}t + \frac{1}{2}\sum_{i,j,k,l} \mathbb{E}\big[(\partial_{ij}\partial_{kl}F)(H)\,\mathrm{d}\langle H_{ij}, H_{kl}\rangle\big].$$

By definition of R we have $R_{iN} = \frac{1}{\sqrt{N}}$ for all i, so that (4.11) yields

$$
\begin{aligned}
\mathrm{d}\langle H_{ij}, H_{kl} \rangle &= \frac{1}{N} \sum_{m,n=1}^{N-1} \left(R_{im} R_{jn} R_{km} R_{ln} + R_{im} R_{jn} R_{kn} R_{lm} \right) \mathrm{d}t \\
&= \frac{1}{N} \left(\delta_{ik} - \frac{1}{N} \right) \left(\delta_{jl} - \frac{1}{N} \right) \mathrm{d}t + \frac{1}{N} \left(\delta_{il} - \frac{1}{N} \right) \left(\delta_{jk} - \frac{1}{N} \right) \mathrm{d}t .
\end{aligned}
$$

Thus, for any $F \in \mathcal{C}^2(\mathcal{M})$ we have (4.10) with

$$
(4.12) \qquad L = \frac{1}{N^3} \sum_{i,j,k,l} \partial_{ij}(\partial_{ij} + \partial_{kl} - \partial_{il} - \partial_{jk}) - \frac{1}{2} \sum_{i,j} H_{ij} \partial_{ij} .
$$

Finally, using $\sum_j H_{ij} = \sum_j H_{ji} = 0$ for $H \in \mathcal{M}$, we observe that L from (4.12) can be rewritten as (4.9). $\qquad \square$

4.1.3. *Outline of proof of Theorems 1.3–1.4.*

Theorems 1.3–1.4 are an immediate consequence of the following two propositions. As in [19], we set

$$
(4.13) \qquad D := d \wedge \frac{N^2}{d^3} .
$$

We always assume $d \in [N^\alpha, N^{2/3-\alpha}]$, which implies $D \geqslant N^\alpha$. To state the two propositions concisely, we introduce the following definition. It will also be convenient in the proofs.

Definition 4.5. *Given $H \in \mathcal{M}$, we denote by $\lambda_1 \geqslant \cdots \geqslant \lambda_{N-1}$ the eigenvalues of $H|_{\mathbf{e}^\perp}$. Consider two random matrix ensembles H_1 and H_2 in $\mathcal{M}$. Then we say that*

> (i) *the* bulk eigenvalue gap statistics *of H_1 and H_2 coincide if for any $n \in \mathbb{N}$, $\phi \in C_c^\infty(\mathbb{R}^n)$, and $\kappa > 0$, we have*

$$
(4.14) \qquad \left(\mathbb{E}_{H_1} - \mathbb{E}_{H_2} \right) \phi \left(N \rho_{sc}(\gamma_i)(\lambda_i - \lambda_{i+1}), \ldots, N \rho_{sc}(\gamma_i)(\lambda_i - \lambda_{i+n}) \right) = \mathrm{o}(1)
$$

> *as $N \to \infty$, uniformly in $i \in [\![\kappa N, (1-\kappa)N]\!]$;*

(ii) the averaged bulk eigenvalue correlation functions of H_1 and H_2 coincide *if for any $n \in \mathbb{N}$, $\phi \in C_c^\infty(\mathbb{R}^n)$, $c > 0$ small enough, and $E \in (-2, 2)$, we have*

(4.15)
$$\int_{\mathbb{R}^n} \phi(x_1, \ldots, x_n) N^n \big(p_{H_1}^{(n)} - p_{H_2}^{(n)} \big) \left(E + \frac{\mathrm{d}x_1}{N\rho_{sc}(E)}, \ldots, E + \frac{\mathrm{d}x_n}{N\rho_{sc}(E)} \right) = \mathrm{o}(1),$$

where the correlation functions $p_{H_i}^{(n)}$ are defined as in (1.8).

Moreover, we say that the bulk eigenvalue statistics of H_1 and H_2 coincide *if (i) and (ii) hold.*

Proposition 4.6. *For any fixed $\delta > 0$ and $t \leqslant N^{-1-\delta} D^{1/2}$, the bulk eigenvalue statistics of $H(0)$ and $H(t)$ coincide.*

Proposition 4.7. *For any fixed $\delta > 0$ and $t \geqslant N^{-1+\delta}$, the bulk eigenvalue statistics of $H(t)$ and $H(\infty) \overset{d}{=} W$ coincide.*

Propositions 4.6–4.7 are proved in Section 4.4. As mentioned in Section 1.1, our main effort and novelty is in proving Proposition 4.6. Proposition 4.7 is essentially a consequence of general results on universality of local eigenvalue statistics with small Gaussian component [63, 62]. The local semicircle law of [19] is an important input in the proofs of both propositions.

Proof of Theorem 1.4. The proof is immediate from Propositions 4.6–4.7, with $\delta \leqslant \alpha/4$. $\qquad\square$

4.2. **Switchings and short-time comparision.** The main result of this section is Proposition 4.8 below. To state it, we introduce the following Sobolev-type semi-norms, whereby the derivatives are taken in the directions of all switchings

(4.16)
$$\mathcal{X} := \big\{ \xi_{ij}^{kl} \in \mathbb{R}^{N \times N} : i, j, k, l \in [\![1, N]\!] \big\}.$$

First, for $r \geqslant 1$, we define an L^r-seminorm on $\mathcal{C}^0(\mathcal{M})$ through

$$(4.17) \qquad \|F\|_{r,t} := \left(\mathbb{E}|F(H(t))|^r\right)^{1/r}.$$

Then, we extend this seminorm to include derivatives: for $F \in \mathcal{C}^n(\mathcal{M})$ we define

$$(4.18) \qquad \|\partial^n F\|_{r,t} := \left\| \sup_{\theta \in [0,1]^n} \sup_{X \in \mathcal{X}^n} \left|\partial_{X_1} \cdots \partial_{X_n} F(\,\cdot\, + (d-1)^{-1/2}\, \theta \cdot X))\right| \right\|_{r,t},$$

where ∂_Y denotes the directional derivative in the direction Y, and for $\theta \in [0,1]^n$ and $X \in \mathcal{X}^n$ we abbreviate

$$\theta \cdot X := \theta_1 X_1 + \cdots + \theta_n X_n.$$

Proposition 4.8. *Let $H(t)$ be the constrained Dyson Brownian motion from Definition 4.2 with initial condition (4.4). Fix $\varepsilon > 0$ and let $r \equiv r(\varepsilon)$ be large enough depending on ε. Then for any $F \in \mathcal{C}^4(\mathcal{M})$ we have*

$$(4.19) \qquad \mathbb{E}F(H(t)) - \mathbb{E}F(H(0)) = O\left(D^{-1/2} N^{1+\varepsilon} \max_{1 \leqslant i \leqslant 4} \int_0^t \|\partial^i F\|_{r,s}\, \mathrm{d}s\right).$$

In the applications in Section 4.4, we will use functions F satisfying $\|\partial^i F\|_{r,s} \leqslant N^c$ for $i \leqslant 4$ and a constant $c > 0$ that can be chosen arbitrarily small. Thus, for $t \leqslant N^{-1-\delta} D^{1/2}$ the right-hand side of (4.19) will be $O(N^{-\delta+\varepsilon+c})$ which is $\emptyset(1)$ provided that $c + \varepsilon < \delta$.

The starting point for the proof of Proposition 4.8 is the idea of [24, Lemma A.1], namely to estimate the left-hand side of (4.19) by estimating $\mathbb{E}(LF(H(t)))$. However, since the entries of $H(t)$ are not independent, a different approach from [24] is needed to control $\mathbb{E}(LF(H(t)))$. We do this by approximating the constrained DBM by a Markovian jump process induced by switchings. This process is defined as follows.

4.2.1. *Switching dynamics.* We introduce a Markovian jump process on simple regular graphs by defining its generator

$$(4.20) \qquad Qf(A) \; := \; \frac{1}{8Nd} \sum_{i,j,m,n} I_{ij}^{mn}(A)\Big(f(A - \xi_{ij}^{mn}) - f(A)\Big),$$

where we recall the definition of a switching from (4.5) and introduce the indicator function

$$(4.21) \qquad I_{ij}^{mn}(A) \; := \; A_{ij}A_{mn}(1 - A_{im})(1 - A_{in})(1 - A_{jm})(1 - A_{jn}).$$

The indicator function $I_{ij}^{mn}(A)$ ensures that the graph encoded by A contains the edges $\{i, j\}$ and $\{m, n\}$ but no other edges between the four vertices $\{i, j, m, n\}$ (i.e. its restriction to $\{i, j, m, n\}$ is 1-regular).

Thus, the process generated by Q is a Markovian jump process whose jump times are the events of a Poisson clock with a constant rate; at each event of the clock, four vertices are selected uniformly at random, and a switching as in Figure 12 is performed on the graph if the four vertices are connected by exactly two edges. It is not hard to show that the uniform measure on d-regular graphs is invariant under this jump process.

Proposition 4.9. *The uniform measure on simple d-regular graphs is invariant under Q. This means that for any function f on the set of simple d-regular graphs we have $\mathbb{E}(Qf(A)) = 0$.*

The proof of the proposition is given in Section 4.2.2, in a slightly more general context. The following proposition shows that the switching jump process generated by Q is well approximated by the constrained DBM generated by L.

The generator L acts naturally on functions of H (denoted henceforth by an uppercase F), and the generator Q on functions of A (denoted henceforth by a lowercase

f). It is therefore convenient to introduce, for any $F \in \mathcal{C}^n(\mathcal{M})$, the abbreviations

$$(4.22) \qquad H = H_A := \frac{1}{\sqrt{d-1}}(A - d\mathbf{e}\mathbf{e}^*), \qquad f(A) = f_F(A) := F(H_A).$$

Proposition 4.10. *For any $F \in \mathcal{C}^4(\mathcal{M})$ and using the notation (4.22) we have*

$$(4.23) \qquad\qquad\qquad Qf(A) = LF(H) + R,$$

where

$$(4.24) \qquad\qquad\qquad \mathbb{E}R = O(D^{-1/2}N^{1+\varepsilon}) \max_{1 \leqslant i \leqslant 4} \|\partial^i F\|_{r,0}.$$

Here $\mathbb{E}$ denotes expectation with respect to the uniform measure on random d-regular graphs A.

The proof of this proposition is also deferred to Section 4.2.2 below. Roughly, the idea of the proof is as follows. By Taylor expansion, we obtain

$$(4.25) \qquad Qf(A) \approx \frac{1}{8Nd} \sum_{i,j,m,n} A_{ij}A_{mn}\left(-\partial_{ij}^{mn}f(A) + \frac{1}{2}(\partial_{ij}^{mn})^2 f(A)\right)$$

with high probability. Now $\mathbb{E}A_{ij} = \frac{d}{N}$ if $i \neq j$. By expanding $A_{ij}A_{mn} = (\frac{d}{N} + (A_{ij} - \frac{d}{N}))(\frac{d}{N} + (A_{mn} - \frac{d}{N}))$, and keeping only the leading terms, we find that the right-hand side of (4.25) becomes by $LF(H)$. Here, for the second-order term on the right-hand side of (4.25), the leading term from $A_{ij}A_{mn}$ is $\frac{d^2}{N^2}$; for the first-order term on the right-hand side of (4.25), the leading term from $A_{ij}A_{mn}$ is $\frac{d}{N}(A_{ij} - \frac{d}{N}) + \frac{d}{N}(A_{mn} - \frac{d}{N})$. Further error terms result from the dependence of the entries of the adjacency matrix.

Before giving the proofs of Propositions 4.9–4.10, we deduce Proposition 4.8 from them.

Proof of Proposition 4.8. By (4.10), it suffices to estimate $\mathbb{E}[LF(H(t))]$. By explicit solution of the constrained DBM, $H(t)$, we find for any fixed $t \geqslant 0$ that

$$(4.26) \qquad H(t) \stackrel{\mathrm{d}}{=} \mathrm{e}^{-t/2}H(0) + (1 - \mathrm{e}^{-t})^{1/2}W$$

where W is a copy of the constrained GOE independent of $H(0)$. For the remainder of the proof, we identify the right-hand side with $H(t)$, abbreviate $H \equiv H(0)$, and introduce the two functions

$$F_W(H) = F_H(W) := F\big(\mathrm{e}^{-t/2}H + (1 - \mathrm{e}^{-t})^{1/2}W\big),$$

where the choice of the argument determines the variables on which the generator L acts. We recall the generator L from (4.9),

$$L = \frac{1}{16N^3}\sum_{i,j,k,l}(\partial_{ij}^{kl})^2 - \frac{1}{32N^2}\sum_{i,j,k,l}H_{ij}^{kl}\partial_{ij}^{kl}.$$

From $\partial^2 = (\mathrm{e}^{-t} + (1 - \mathrm{e}^{-t}))\partial^2$, $\mathrm{e}^{-t/2}\partial F = \partial F_W$, and $(1 - \mathrm{e}^{-t})^{1/2}\partial F = \partial F_H$, we then deduce that $LF\big(\mathrm{e}^{-t/2}H + (1 - \mathrm{e}^{-t})^{1/2}W\big) = LF_W(H) + LF_H(W)$. We therefore get

$$\mathbb{E}[LF(H(t))] = \mathbb{E}[LF_W(H)] + \mathbb{E}[LF_H(W)] = \mathbb{E}[LF_W(H)],$$

where in the second step we used that the constrained GOE, W, is invariant with respect to the generator L.

Next, we define $f_W(A) := F_W(H)$ where $H \equiv H_A$ is defined as (4.22). By Proposition 4.9, the random d-regular graph A is invariant with respect to the generator Q, and Proposition 4.10 therefore yields

$$\mathbb{E}[LF_W(H)] = \mathbb{E}[Qf_W(A)] + \mathrm{O}(D^{-1/2}N^{1+\varepsilon})\max_{1 \leqslant i \leqslant 4}\|\partial^i F_W\|_{r,0}$$

$$= \mathrm{O}(D^{-1/2}N^{1+\varepsilon})\max_{1 \leqslant i \leqslant 4}\|\partial^i F\|_{r,t}.$$

FIGURE 13. Given four vertices i, j, k, l with two edges between them, there are two possible switchings. By equipping the edges with directions, one of these two switchings can be selected canonically.

Thus, with (4.10), we have shown that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[F(H(t))] = \mathrm{O}(D^{-1/2}N^{1+\varepsilon})\max_{1\leqslant i\leqslant 4}\|\partial^i F\|_{r,t},$$

and the claim follows by integrating over t. $\qquad\square$

4.2.2. *Proofs of Propositions 4.9–4.10.* Propositions 4.9–4.10 concern switchings of regular graphs. Switchings played an important role in the proof of the local semicircle law for random regular graphs [19]. Here we use simple switchings instead of the double switchings needed in [19].

Given two disjoint edges of a regular graph such that the graph has no other edges between the vertices incident to these two edges, there are two possible switchings; see Figure 13. To specify one of these two switchings, it is convenient to assign to each of the edges to be switched a direction; there is then a canonical choice between the two possible switchings. We write ij for the edge $\{i, j\}$ directed from i to j.

We consider sets S of two directed edges of the complete graph, which we write in the form $S = \{ij, kl\}$. We denote by $[S] = \{i, j, k, l\}$ the set of vertices incident to the edges of S. For two such sets S and S', we define the indicator functions

$$(4.27) \qquad\qquad I(S) \equiv I(S; A) := \mathbf{1}(|[S]| = 4 \text{ and } E|_{[S]} \text{ is 1-regular}),$$

$$(4.28) \qquad J(S, S') \equiv J(S, S'; A) := \mathbf{1}([S] \cap [S'] = \emptyset),$$

where $E \equiv E(A) := \{\{i, j\} : A_{ij} = 1\}$ is the set of (undirected) edges of the graph encoded by A, and $E|_B := \{e \in E : e \subset B\}$ is the restriction of the graph E to the

subset of vertices B. The indicator functions are illustrated in Figure 14. Note that $I_{ij}^{mn} = A_{ij} A_{mn} I(\{ij, mn\})$ (recall (4.21)).

Remark 4.11. *The definitions* (4.27)–(4.28) *are similar to those given in* [19, Section 6], *with the following differences. First, the current set S consists of two directed edges instead of the three undirected edges in* [19]. *Because of the directions contained in the current set S, it effectively incorporates the extra parameter s of* [19, Section 6]. *Second, the edges in S are edges of the complete graph, and we do not assume that they are contained in some regular graph A; we will ultimately define the switching associated with the set S to act trivially unless S is contained in the edges E of the given graph.*

For a set $S = \{ij, kl\}$ of two directed edges, we define the *switching*

$$(4.29) \qquad T_S(A) := \begin{cases} A - \xi_{ij}^{kl} & \text{if } I(S) = 1, A_{ij} = 1, A_{kl} = 1 \\ A + \xi_{ij}^{kl} & \text{if } I(S) = 1, A_{ik} = 1, A_{jl} = 1 \\ A & \text{otherwise}, \end{cases}$$

where we recall the definition of ξ_{ij}^{kl} from (4.5). In words, $T_S(A)$ switches the edges S if they are contained in A and are switchable in the sense that the switching results again in a d-regular graph. Moreover, for S, S' as above, we define

$$(4.30) \qquad T_{S,S'}(A) := \begin{cases} T_{S'}(T_S(A)) & \text{if } J(S, S') = 1 \\ A & \text{otherwise}. \end{cases}$$

In words, $T_{S,S'}(A)$ switches the edges in S and S' if they are contained in A and the two switchings do not interfere with each other.

Lemma 4.12. *For any fixed S, S' we have $A \stackrel{d}{=} T_S(A)$ and $A \stackrel{d}{=} T_{S,S'}(A)$.*
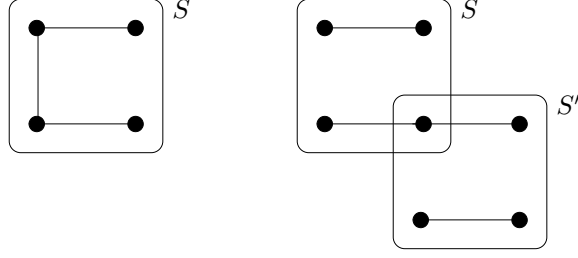
FIGURE 14. In the left diagram, $I(S) = 0$ since the restricted graph is not 1-regular. In the right diagram, $J(S, S') = 0$ since the two sets of vertices intersect.

Proof. It is easy to check that $T_S(A)$ is a d-regular graph if and only if A is. Moreover, $T_S(T_S(A)) = A$, so T_S is a bijection on the set of d-regular graphs. Since the distribution of A is uniform, we obtain $A \stackrel{\mathrm{d}}{=} T_S(A)$. The second claim follows similarly from $T_{S,S'}(T_{S,S'}(A)) = A$. $\qquad\square$

Now Proposition 4.9 follows easily.

Proof of Proposition 4.9. For any f, we get

$$
\begin{aligned}
\sum_{i,j,m,n} \mathbb{E}\big(I_{ij}^{mn}(A)f(A)\big) &= \sum_{i,j,m,n} \mathbb{E}\big(A_{ij}A_{mn}I(\{ij, mn\}; A)f(A)\big) \\
&= \sum_{i,j,m,n} \mathbb{E}(A_{im}A_{jn}I(\{ij, mn\}; A + \xi_{ij}^{mn})f(A + \xi_{ij}^{mn})) \\
&= \sum_{i,j,m,n} \mathbb{E}\big(A_{ij}A_{mn}I(\{ij, mn\}; A)f(A - \xi_{ij}^{mn})\big) \\
&= \sum_{i,j,m,n} \mathbb{E}(I_{ij}^{mn}(A)f(A - \xi_{ij}^{mn})),
\end{aligned}
$$

where the first and last steps follows from the definition of I_{ij}^{mn}, the second step from Lemma 4.12, and the third step from the exchangability of i, j, m, n and using $I(S; A) = I(S; T_S(A))$. This concludes the proof. $\qquad\square$

For the proof of Proposition 4.10 we shall need estimates on the moments of entries of the adjacency matrix, as well as estimates on such moments restricted to

low-probability events where the indicator functions (4.27)–(4.28) are zero. These estimates are collected in the following sequence of lemmas.

The following two lemmas show that moments of the entries of the adjacency matrix behave roughly like those of an Erdős-Rényi graph.

Lemma 4.13. *Let $b \ll N$ and $i_1, j_1, \ldots, i_b, j_b \in [\![1, N]\!]$. Then for any $p \in [\![1, N]\!]$ and $q \in [\![1, N]\!] \setminus \{i_1, j_1, \ldots, i_b, j_b\}$, we have*

$$(4.31) \qquad \mathbb{E}(A_{i_1 j_1} \cdots A_{i_b j_b} A_{pq}) \;=\; O\Big(\frac{d}{N}\Big) \mathbb{E}(A_{i_1 j_1} \cdots A_{i_b j_b}),$$

where we use the convention $\mathbb{E}(A_{i_1 j_1} \cdots A_{i_b j_b}) = 1$ if $b = 0$.

Proof. Set $X := A_{i_1 j_1} \cdots A_{i_b j_b}$ and $I := \{i_1, j_1, \ldots, i_b, j_b, p\}$. Then, since $\sum_n A_{pn} = d$ for any p, we find for any $q \notin I$ that

$$\mathbb{E}(X) \;=\; \frac{1}{d} \sum_n \mathbb{E}(X A_{pn}) \;=\; \frac{1}{d} \sum_{n \notin I} \mathbb{E}(X A_{pn}) + \frac{1}{d} \sum_{n \in I} \mathbb{E}(X A_{pn})$$

$$\geqslant \; \frac{1}{d} \sum_{n \notin I} \mathbb{E}(X A_{pn}) \;=\; \frac{N - |I|}{d} \mathbb{E}(X A_{pq}),$$

where in the third step we used that $X A_{pn} \geqslant 0$ and in the last step that the law of A is invariant under permutation of vertices. Using that $|I| \leqslant N/2$ by assumption on b, the claim now follows. $\qquad \square$

As a consequence of Lemma 4.13, we obtain the following explicit bounds.

Lemma 4.14. *Suppose that $|\{i, j, m, n\}| = 4 - a$ and $|\{i, j, k, l, m, n, p, q\}| = 8 - b$. Then*

$$(4.32) \qquad \mathbb{E}(A_{ij} A_{mn}) \;=\; O\Big(\frac{d}{N}\Big)^{2 - \lfloor a/2 \rfloor},$$

$$(4.33) \qquad \mathbb{E}(A_{ij} A_{mn} A_{kl} A_{pq}) \;=\; O\Big(\frac{d}{N}\Big)^{4 - \lfloor b/2 \rfloor}.$$

Proof. Since $A_{ss} = 0$ for all s, we can assume that $i \neq j$, $m \neq n$, $k \neq l$, and $p \neq q$, and thus $a \leqslant 2$ and $b \leqslant 4$. Then (4.32)–(4.33) follow easily from Lemma 4.13. $\qquad\square$

In the next two lemmas, we estimate moments restricted to low-probability events where the indicator functions (4.27)–(4.28) vanish, i.e. we estimate the contribution of graphs A that are not switchable. Throughout the rest of this section, for given indices i, j, k, l, m, n, p, q we use the abbreviations

$$(4.34) \qquad I_1 := I(\{ij, mn\}; A), \quad I_2 := I(\{kl, pq\}; A),$$

$$(4.35) \qquad J_{12} := J(\{ij, mn\}, \{kl, pq\}; A), \quad I_{12} := I_1 I_2 J_{12},$$

with I and J defined in (4.27)–(4.28).

Lemma 4.15. *Let* $|\{i, j, m, n\}| = 4 - a$ *and* $|\{i, j, k, l, m, n, p, q\}| = 8 - b$. *Then*

$$(4.36) \qquad \mathbb{E}((A_{ij}A_{mn} + A_{im}A_{jn})(1 - I_1)) = O\left(\frac{d}{N}\right)^{3-a}.$$

$$(4.37) \qquad \mathbb{E}((A_{ij}A_{mn} + A_{im}A_{jn})(A_{kl}A_{pq} + A_{kp}A_{lq})(1 - I_{12})) = O\left(\frac{d}{N}\right)^{5-b}.$$

Proof. First, assume that i, j, k, l, m, n, p, q are all distinct, i.e. we consider the case $a = b = 0$. Then, since $|\{i, j, m, n\}| = 4$ and $I_1 = 0$ implies that the graph A restricted to $\{i, j, m, n\}$ is not 1-regular, we find

$$\mathbb{E}(A_{ij}A_{mn}(1 - I_1)) \leqslant \mathbb{E}(A_{ij}A_{mn}(A_{im} + A_{in} + A_{jm} + A_{jn})),$$

$$\mathbb{E}(A_{im}A_{jn}(1 - I_1)) \leqslant \mathbb{E}(A_{im}A_{jn}(A_{ij} + A_{mn} + A_{in} + A_{jm})),$$

and Lemma 4.13 implies that the right-hand sides are bounded by $O(d/N)^3$. The proof of (4.37) for $b = 0$ is analogous. We only consider the term $A_{ij}A_{kl}A_{mn}A_{pq}$; the others dealt with similarly. First, note that $J_{12} = 1$ if $|\{i, j, k, l, m, n, p, q\}| = 8$. Since $|\{i, j, k, l, m, n, p, q\}| = 8$ and $I_1 I_2 = 0$ imply that $E|_{\{i,j,m,n\}}$ or $E|_{\{k,l,p,q\}}$ has at

least three edges, we find

$$\mathbb{E}\big((A_{ij}A_{mn}A_{kl}A_{pq}(1 - I_1 I_2 J_{12})\big) = \mathbb{E}\big((A_{ij}A_{mn}A_{kl}A_{pq}(1 - I_1 I_2)\big)$$

$$\leqslant \mathbb{E}\big((A_{ij}A_{mn}A_{kl}A_{pq}(A_{im} + A_{in} + A_{jm} + A_{jn} + A_{kp} + A_{kq} + A_{lp} + A_{lq})\big) = O\Big(\frac{d}{N}\Big)^5,$$

where the last step follows from Lemma 4.13.

Finally, if $a > 0$ we have $I_1 = 0$, and if $b > 0$ we have $I_{12} = 0$. In these cases, we can directly apply (4.32) and (4.33), respectively, and the claim follows since $2 - \lfloor a/2 \rfloor \geqslant 3 - a$ if $a > 0$ and $4 - \lfloor b/2 \rfloor \geqslant 5 - b$ if $b > 0$. □

As a consequence of Lemma 4.15, we obtain the following averaged estimates.

Lemma 4.16. *If $|\{i, j\}| = 2 - a$ and $|\{i, j, k, l\}| = 4 - b$, then*

$$(4.38) \qquad \frac{1}{N^2} \sum_{m,n} \mathbb{E}\big((A_{ij}A_{mn} + A_{im}A_{jn})(1 - I_1)\big) = O\Big(\frac{d}{N}\Big)^{3-a},$$

$$(4.39)$$
$$\frac{1}{N^4} \sum_{m,n} \sum_{p,q} \mathbb{E}\big((A_{ij}A_{mn} + A_{im}A_{jn})(A_{kl}A_{pq} + A_{kp}A_{lq})(1 - I_{12})\big) = O\Big(\frac{d}{N}\Big)^{5-b}.$$

Moreover,

$$(4.40) \qquad \frac{1}{N^4} \sum_{i,j,m,n} \mathbb{E}\big((A_{ij}A_{mn} + A_{im}A_{jn})(1 - I_1)\big) = O\Big(\frac{d}{N}\Big)^3,$$

$$(4.41)$$
$$\frac{1}{N^8} \sum_{i,j,m,n} \sum_{k,l,p,q} \mathbb{E}\big((A_{ij}A_{mn} + A_{im}A_{jn})(A_{kl}A_{pq} + A_{kp}A_{lq})(1 - I_{12})\big) = O\Big(\frac{d}{N}\Big)^5.$$

Proof. To prove (4.38), we split the summation over m, n by fixing $|\{i, j, m, n\}| = 4 - a - s$ where $s \in [\![0, 2]\!]$; there are $O(N^{2-s})$ terms corresponding to each $s \in [\![0, 2]\!]$. By (4.36), the left-hand side of (4.38) is bounded by

$$O\Big(\frac{d}{N}\Big)^{3-a} + \sum_{s=1}^{2} O(N^{-s})O\Big(\frac{d}{N}\Big)^{3-a-s} = O\Big(\frac{d}{N}\Big)^{3-a}.$$

The proofs of (4.39)–(4.41) are analogous. $\qquad\square$

Finally, as a consequence of Lemmas 4.13–4.16 and the Hölder inequality, we obtain the following estimates incorporating an arbitrary function $f(A)$. These and the remainder of the proof of Proposition 4.10 are simplest to state in terms of versions of the seminorms (4.17)–(4.18) for $t = 0$ without rescaling by $(d-1)^{-1/2}$. Thus, instead of (4.17) and (4.18), we use the seminorms

$$\|f\|_r := \left(\mathbb{E}|f(A)|^r\right)^{1/r}$$

and

$$\|\partial^n f\|_r := \left\|\sup_{\theta\in[0,1]^n} \sup_{X\in\mathcal{X}^n} \left|\partial_{X_1}\cdots\partial_{X_n} f(\,\cdot\, + \theta\cdot X)\right|\right\|_r.$$

Lemma 4.17. *Fix $\varepsilon > 0$ and let $r \equiv r(\varepsilon)$ be large enough depending on ε. Let $f \in \mathcal{C}^0(\mathcal{M})$ satisfy $\|f\|_r \leqslant 1$. Then if $|\{i,j\}| = 2-a$ and $|\{i,j,k,l\}| = 4-b$, we have*

$$(4.42) \qquad \frac{1}{N^2}\sum_{m,n}\mathbb{E}\left(A_{ij}A_{mn}f(A)\right) = O\left(\frac{d}{N}\right)^{2-\lfloor a/2\rfloor-\varepsilon},$$

$$(4.43) \qquad \frac{1}{N^4}\sum_{m,n,p,q}\mathbb{E}\left(A_{ij}A_{mn}A_{kl}A_{pq}f(A)\right) = O\left(\frac{d}{N}\right)^{4-\lfloor b/2\rfloor-\varepsilon},$$

$$(4.44) \qquad \frac{1}{N^2}\sum_{m,n}\mathbb{E}\left((A_{ij}A_{mn} + A_{im}A_{jn})\bar{I}_1 f(A)\right) = O\left(\frac{d}{N}\right)^{3-a-\varepsilon},$$

(4.45)
$$\frac{1}{N^4}\sum_{m,n}\sum_{p,q}\mathbb{E}\left((A_{ij}A_{mn} + A_{im}A_{jn})(A_{kl}A_{pq} + A_{kp}A_{lq})\bar{I}_{12} f(A)\right) = O\left(\frac{d}{N}\right)^{5-b-\varepsilon},$$

$$(4.46) \qquad \frac{1}{N^4}\sum_{i,j,m,n}\mathbb{E}\left((A_{ij}A_{mn} + A_{im}A_{jn})\bar{I}_1 f(A)\right) = O\left(\frac{d}{N}\right)^{3-\varepsilon},$$

(4.47)
$$\frac{1}{N^8}\sum_{i,j,m,n}\sum_{k,l,p,q}\mathbb{E}\left((A_{ij}A_{mn} + A_{im}A_{jn})(A_{kl}A_{pq} + A_{kp}A_{lq})\bar{I}_{12} f(A)\right) = O\left(\frac{d}{N}\right)^{5-\varepsilon},$$

where $\bar{I}_1 := 1 - I_1$, $\bar{I}_{12} := 1 - I_{12}$, and the indicator functions I_1 and I_{12} were defined in (4.34)–(4.35).

Proof. We only prove (4.46); the other estimates are proved similarly and we comment on the differences at the end of the proof. By Hölder's inequality, applied twice, first to $\mathbb{E}(\cdot)$ and then to the sum over m, n, we obtain from (4.40) that

$$\frac{1}{N^4} \sum_{i,j,m,n} \mathbb{E}\big((A_{ij}A_{mn} + A_{im}A_{jn})(1 - I_1)f(A)\big)$$

$$\leqslant \frac{1}{N^4} \sum_{i,j,m,n} \big[\mathbb{E}\big((A_{ij}A_{mn} + A_{im}A_{jn})(1 - I_1)\big)\big]^{1-1/r} \|f\|_r$$

$$\leqslant \Big(\frac{1}{N^4} \sum_{i,j,m,n} \mathbb{E}((A_{ij}A_{mn} + A_{im}A_{jn})(1 - I_1))\Big)^{1-1/r} \|f\|_r$$

$$\leqslant O\Big(\frac{d}{N}\Big)^{3-3/r} \|f\|_r = O\Big(\frac{d}{N}\Big)^{3-\varepsilon} \|f\|_r,$$

where we chose r large enough that $3/r \leqslant \varepsilon$.

To prove (4.47), we use (4.41) instead of (4.40), and to prove (4.42)–(4.43) we apply (4.31) instead of (4.36). To prove (4.44)–(4.45), we use (4.38)–(4.39). This concludes the proof. $\qquad\square$

The next lemma estimates the effect of replacing A_{ij} by its mean d/N, or, equivalently, of conditioning on $\{A_{ij} = 1\}$. Since the entries of A are not independent, we use switchings to analyse such a conditioning.

Lemma 4.18. *Fix $\varepsilon > 0$ and let $r \equiv r(\varepsilon)$ be large enough depending on ε. For any $f \in \mathcal{C}^2(\mathcal{M})$ and any i, j, k, l with $|\{i, j\}| = 2 - a$ and $|\{i, j, k, l\}| = 4 - b$, we have*

$$(4.48) \qquad \mathbb{E}\Big(f(A)\Big(A_{ij} - \frac{d}{N}\Big)\Big) = O\Big(\frac{d}{N}\Big)^{1-\varepsilon} \|\partial f\|_r + O\Big(\frac{d}{N}\Big)^{2-a-\varepsilon} \|f\|_r,$$

$$(4.49) \quad \mathbb{E}\Big(f(A)\Big(A_{ij} - \frac{d}{N}\Big)\Big(A_{kl} - \frac{d}{N}\Big)\Big) = O\Big(\frac{d}{N}\Big)^{2-\varepsilon} \|\partial^2 f\|_r + O\Big(\frac{d}{N}\Big)^{3-b-\varepsilon} \|f\|_r.$$

Proof. We begin with (4.48). Since $A \in \mathcal{M} + d\mathbf{e}\mathbf{e}^*$, we have $\sum_{m,n} A_{mn} = Nd$ and $\sum_m A_{im} = \sum_n A_{jn} = d$ for all i, and the left-hand side of (4.48) is therefore equal to

$$
(4.50) \qquad \mathbb{E}\left(f(A)\left(A_{ij} - \frac{d}{N}\right)\right) = \frac{1}{Nd}\sum_{m,n} \mathbb{E}\left(f(A)(A_{ij}A_{mn} - A_{im}A_{jn})\right).
$$

Using (4.44), using the notation from (4.34), we therefore find

$$
\mathbb{E}\left(f(A)\left(A_{ij} - \frac{d}{N}\right)\right) = \frac{1}{Nd}\sum_{m,n} \mathbb{E}\left(f(A)(A_{ij}A_{mn} - A_{im}A_{jn})I_1\right) + O\left(\frac{d}{N}\right)^{2-a-\varepsilon}\|f\|_r.
$$

Because of the indicator function I_1, the first term on the right-hand side vanishes unless $a = 0$. Therefore we may assume that $a = 0$ when estimating it. By Lemma 4.12, and since $I_1(A) = I_1(T_S(A))$ with $S = \{ij, mn\}$, the first term on the right-hand side equals

$$
(4.51) \qquad \frac{1}{Nd}\sum_{m,n} \mathbb{E}\left(\left(f(A) - f(A - \xi_{ij}^{mn})\right)A_{ij}A_{mn}I_1\right).
$$

The difference of the f's is bounded in absolute value by $\sup_{\theta \in [0,1]} \sup_{X \in \mathcal{X}} |\partial_X f(A + \theta X)|$. Hence, (4.42) implies that (4.51) is bounded by

$$
O\left(\frac{d}{N}\right)^{1-\varepsilon}\|\partial f\|_r.
$$

This concludes the proof of (4.48).

The proof of (4.49) is similar. As in (4.50), we write

$$
\left(A_{ij} - \frac{d}{N}\right)\left(A_{kl} - \frac{d}{N}\right) = \frac{1}{(Nd)^2}\sum_{m,n,p,q}(A_{ij}A_{mn} - A_{im}A_{jn})(A_{kl}A_{pq} - A_{kp}A_{lq}).
$$

As above, we write $1 = I_{12} + (1 - I_{12})$ inside the expectation on the left-hand side of (4.49). The second term yields a contribution of order $O(\frac{d}{N})^{3-b-\varepsilon}\|f\|_r$, by (4.45). The first term is zero unless $b = 0$ because of the factor J_{12} in I_{12}. We may therefore assume that $b = 0$ for the estimate of the first term. Using Lemma 4.12, as in (4.51),

we find that the first term is equal to

(4.52)
$$\frac{1}{(Nd)^2} \sum_{m,n,p,q} \mathbb{E}\left(\left(f(A) - f(A - \xi_{ij}^{mn}) - f(A - \xi_{kl}^{pq}) + f(A - \xi_{ij}^{mn} - \xi_{kl}^{pq}) \right) A_{ij} A_{mn} A_{kl} A_{pq} I_{12} \right).$$

The difference of the four f's is bounded in absolute value by

$$\sup_{\theta_1,\theta_2 \in [0,1]} \sup_{X_1,X_2 \in \mathcal{X}} \left| \partial_{X_1} \partial_{X_2} f(A + \theta_1 X_1 + \theta_2 X_2) \right|.$$

By (4.43), we therefore find that (4.52) is bounded in absolute value by

$$O\left(\frac{d}{N}\right)^{2-\varepsilon} \|\partial^2 f\|_r.$$

This concludes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Finally, with the preparations provided by the previous lemmas, we now complete the proof of Proposition 4.10.

Proof of Proposition 4.10. First note that $I_{ij}^{mn} = A_{ij} A_{mn} I_1$. By Taylor expansion, and writing $I_1 = 1 + (I_1 - 1)$, we therefore have

$$(4.53) \quad Qf(A) \; = \; \frac{1}{8Nd} \sum_{i,j,m,n} A_{ij} A_{mn} \left(-\partial_{ij}^{mn} f(A) + \frac{1}{2} (\partial_{ij}^{mn})^2 f(A) \right) + N^2 (R_1 + R_2),$$

where

$$R_1 \; = \; O\left(\frac{N}{d}\right) \frac{1}{N^4} \sum_{i,j,m,n} A_{ij} A_{mn} (1 - I_1) \sup_{\theta \in [0,1]} \sup_{X \in \mathcal{X}} |\partial_X f(A + \theta X)|,$$

$$R_2 \; = \; O\left(\frac{N}{d}\right) \frac{1}{N^4} \sum_{i,j,m,n} A_{ij} A_{mn} \sup_{\theta \in [0,1]^3} \sup_{X \in \mathcal{X}^3} |\partial_{X_1} \partial_{X_2} \partial_{X_3} f(A + \theta \cdot X)|.$$

By (4.46) and (4.42), respectively, the two error terms are estimated by

$$\mathbb{E} R_1 \; = \; O\left(\frac{d}{N}\right)^{2-\varepsilon} \|\partial f\|_r, \quad \mathbb{E} R_2 \; = \; O\left(\frac{d}{N}\right)^{1-\varepsilon} \|\partial^3 f\|_r.$$

Next, we estimate the main terms in (4.53), which we write as

$$(4.54) \qquad \frac{1}{8Nd} \sum_{i,j,k,l} A_{ij} A_{kl} \left(-\partial_{ij}^{kl} f(A) + \frac{1}{2} (\partial_{ij}^{kl})^2 f(A) \right).$$

The idea is to write $A_{ij} = \frac{d}{N} + (A_{ij} - \frac{d}{N})$ and likewise for A_{kl}. For the second-order term in (4.54), the term obtained by selecting both factors $\frac{d}{N}$ yields the main contribution. More precisely, we write

$$\frac{1}{16Nd} \sum_{i,j,k,l} A_{ij} A_{kl} (\partial_{ij}^{kl})^2 f(A) = \frac{d}{16N^3} \sum_{i,j,k,l} (\partial_{ij}^{kl})^2 f(A) + N^2 (R_3 + R_4),$$

where

$$R_3 = \frac{N}{8d} \frac{1}{N^4} \sum_{i,j,k,l} \left(((\partial_{ij}^{kl})^2 f(A)) \left(A_{ij} - \frac{d}{N} \right) \frac{d}{N} \right),$$

$$R_4 = \frac{N}{16d} \frac{1}{N^4} \sum_{i,j,k,l} \left(((\partial_{ij}^{kl})^2 f(A)) \left(A_{ij} - \frac{d}{N} \right) \left(A_{kl} - \frac{d}{N} \right) \right).$$

By (4.48) and (4.49), respectively, with f replaced by $(\partial_{ij}^{kl})^2 f$, we obtain

$$\mathbb{E}(R_3 + R_4) = O\left(\frac{d}{N} \right)^{1-\varepsilon} \left(\|\partial^3 f\|_r + \|\partial^4 f\|_r \right) + O\left(\frac{d}{N} \right)^{2-\varepsilon} \|\partial^2 f\|_r.$$

Next, we estimate the first-order term in (4.54) using a similar argument. Here the term obtained by selecting both factors $\frac{d}{N}$ from A_{ij} and A_{kl} vanishes because $\sum_{i,j,k,l} \partial_{ij}^{kl} = 0$; the main contribution is given by the mixed term. More precisely, we write

$$\frac{1}{8Nd} \sum_{i,j,k,l} A_{ij} A_{kl} \partial_{ij}^{kl} f(A) = \frac{d}{8N^3} \sum_{i,j,k,l} \partial_{ij}^{kl} f(A) + \frac{1}{4N^2} \sum_{i,j,k,l} \left(A_{ij} - \frac{d}{N} \right) \partial_{ij}^{kl} f(A) + N^2 R_5$$

$$= \frac{\sqrt{d-1}}{4N^2} \sum_{i,j,k,l} H_{ij} \partial_{ij}^{kl} f(A) + N^2 R_5$$

$$= \frac{\sqrt{d-1}}{32N^2} \sum_{i,j,k,l} H_{ij}^{kl} \partial_{ij}^{kl} f(A) + N^2 R_5,$$

where

$$R_5 \;=\; \frac{N}{8d}\frac{1}{N^4}\sum_{i,j,k,l}\left((\partial_{ij}^{kl}f(A))\left(A_{ij}-\frac{d}{N}\right)\left(A_{kl}-\frac{d}{N}\right)\right).$$

By (4.49), with f replaced by $\partial_{ij}^{kl}f$, we obtain

$$\mathbb{E}R_5 \;=\; O\left(\frac{d}{N}\right)^{1-\varepsilon}\|\partial^3 f\|_r + O\left(\frac{d}{N}\right)^{2-\varepsilon}\|\partial f\|_r.$$

We conclude that

$$\begin{aligned}
Qf(A) \;&=\; \frac{d}{16N^3}\sum_{i,j,k,l}(\partial_{ij}^{kl})^2 f(A) - \frac{\sqrt{d-1}}{32N^2}\sum_{i,j,k,l}H_{ij}^{kl}\partial_{ij}^{kl}f(A) + N^2\sum_{i=1}^{5}R_i \\
&=\; \frac{d-1}{16N^3}\sum_{i,j,k,l}(\partial_{ij}^{kl})^2 f(A) - \frac{\sqrt{d-1}}{32N^2}\sum_{i,j,k,l}H_{ij}^{kl}\partial_{ij}^{kl}f(A) + N^2\sum_{i=1}^{6}R_i,
\end{aligned}$$

where we defined

$$R_6 \;:=\; \frac{1}{16N}\frac{1}{N^4}\sum_{i,j,k,l}(\partial_{ij}^{kl})^2 f(A).$$

Clearly, $\mathbb{E}R_6 = O\left(\frac{1}{N}\right)\|\partial^2 f\|_r$.

Using the notations introduced in (4.22), we have $\sqrt{d-1}\,\partial f(A) = \partial F(H)$. Hence we obtain (4.23) with $R := N^2\sum_{i=1}^{6}R_i$. The error term R is estimated, using the above estimates on $\mathbb{E}R_i$, as

$$\begin{aligned}
\mathbb{E}R \;&=\; O(N^{2+\varepsilon})\left[\left(\frac{d}{N}\right)^2\left(\|\partial f\|_r + \|\partial^2 f\|_r\right) + \frac{1}{N}\|\partial^2 f\|_r + \frac{d}{N}\left(\|\partial^3 f\|_r + \|\partial^4 f\|_r\right)\right] \\
&=\; O(D^{-1/2}N^{1+\varepsilon})\left[\|\partial F\|_{r,0} + D^{-1/2}\|\partial^2 F\|_{r,0} + \|\partial^3 F\|_{r,0} + D^{-1/2}\|\partial^4 F\|_{r,0}\right],
\end{aligned}$$

as claimed. $\qquad\qquad\square$

4.3. **Stability of eigenvectors and eigenvalues.** In this section we derive basic stability properties for the eigenvalues and eigenvectors of the Dyson Brownian motion $H(t)$. These allow us to deduce estimates on the eigenvalues and eigenvectors of $H(t)$, assuming similar estimates have been proved for $H(0)$.

As discussed in Section 4.1.1, we consider a general Dyson Brownian motion $H(t)$ on an M-dimensional Hilbert space V, with normalization constant N as in (4.1). For the usual DBM we have $N = M$, while for the constrained DBM we have $M = N - 1$; we always assume that N and M are comparable. We denote by $\lambda_1(t) \geqslant \cdots \geqslant \lambda_M(t)$ the eigenvalues of $H(t)$, and by $\mathbf{v}_1(t), \ldots, \mathbf{v}_M(t) \in V$ the associated normalized eigenvectors of $H(t)$. Moreover, we define the Stieltjes transform of the empirical spectral measure of $H(t)$ by $s(t; z) := \frac{1}{M} \sum_{i=1}^{M} \frac{1}{\lambda_i(t) - z}$.

Throughout the rest of the paper, we use the following notion of high probability events and high probability bounds, introduced in [37].

Definition 4.19. *(i) We say that an event Ξ has* high probability *if for every $\zeta > 0$ there exists an $N_0(\zeta) > 0$ such that $\mathbb{P}(\Xi^c) \leqslant N^{-\zeta}$ for $N \geqslant N_0(\zeta)$.*

(ii) For nonnegative random variables A, B, we write $A \prec B$ or $A = O_\prec(B)$ if for any $\zeta > 0$ there exists an $N_0(\zeta)$ such that $\mathbb{P}(A > N^{1/\zeta} B) \leqslant N^{-\zeta}$ for $N \geqslant N_0(\zeta)$.

If the event Ξ from (i) and the random variables A and B from (ii) depend on some additional parameter $u \in U$ in some possibly N-dependent set U, we we say that (i) and (ii) hold uniformly *in u if $N_0(\zeta)$ does not depend on u.*

Throughout the following, the definitions (i) and (ii) will always be uniform in all parameters, such as z, any matrix indices, and deterministic vectors. Note that $\prec$ is compatible with the usual algebraic operations, so that for instance we have $\sum_i A_i \prec \sum_i B_i$ provided that $A_i \prec B_i$ for all i and the size of the index set for i is $N^{O(1)}$.

4.3.1. *Delocalization of eigenvectors.* The following result shows that if all eigenvectors of $H(0)$ are uniformly delocalized in some direction $\mathbf{q} \in V$, then with high probability they remain delocalized in this direction under the DBM on V, for any time $t > 0$.

Lemma 4.20. *Suppose that $H(t)$ is the DBM on an M-dimensional space V. Let $\mathbf{q} \in V$ and suppose that $\max_i |\mathbf{q} \cdot \mathbf{v}_i(0)| \leqslant B$. Then, for any $t > 0$, any $i \in [\![1, M]\!]$, and $\xi \gg 1$,*

$$(4.55) \qquad\qquad \mathbb{P}\left(|\mathbf{q} \cdot \mathbf{v}_i(t)| \geqslant \xi B\right) \leqslant e^{-\frac{1}{2}\xi^2}.$$

In particular,

$$(4.56) \qquad\qquad |\mathbf{q} \cdot \mathbf{v}_i(t)| \prec B.$$

Lemma 4.20 is a simple consequence of the *eigenvector moment flow* (EMF) introduced in [24]. Suppose for simplicity that the eigenvalues of $H(0)$ are distinct. Then the eigenvalue process $(\lambda_i(t))$ is almost surely continuous and simple for all $t > 0$; see [24] for more details. We study the dynamics of the eigenvectors $\mathbf{v}_i(t)$ by conditioning on the eigenvalue process; see again [24] for a precise construction. Hence, for the following argument, we condition on $(\lambda_i(t))$ and regard the eigenvalue process as deterministic.

We give the definition of the EMF restricted to moments of a fixed order $p \in \mathbb{N}$. The configuration space is $\Omega_p := \left\{\eta = (\eta_i)_{i=1}^M \in \mathbb{N}^M : \sum_{i=1}^M \eta_i = p\right\}$. The configurations $\eta \in \Omega_p$ are interpreted as configurations of p particles on the lattice $[\![1, M]\!]$, whereby a single site of $[\![1, M]\!]$ may be occupied by multiple particles. We denote by $\eta^{i,j} := \eta + \mathbf{1}(\eta_i > 0)(\mathbf{e}_j - \mathbf{e}_i)$ the configuration obtained from η by moving one particle from i to j. The time-dependent generator $R(t)$ of the EMF is defined by

$$(R(t)f)(\eta) := \sum_{i \neq j} W_{ij}(t) 2\eta_i(1 + 2\eta_i)(f(\eta^{i,j}) - f(\eta)),$$

where

$$W_{ij}(t) := \frac{1}{N(\lambda_i(t) - \lambda_j(t))^2}.$$

For our purposes, the precise form of the coefficients $W_{ij}(t)$ is not important; we only use that they are nonnegative and continuous in t. The p-particle EMF is given by the equation

$$(4.57) \qquad \partial_t f_t(\eta) = (R(t)f_t)(\eta), \qquad f_0 : \Omega_p \to \mathbb{R} \text{ given}.$$

This is a linear (time-dependent) ODE on a finite dimensional vector space, and thus well-posed. It is also easy to see that it is contractive on $L^\infty(\Omega)$ in the sense that $\|f_t\|_{L^\infty(\Omega_p)} \leqslant \|f_0\|_{L^\infty(\Omega_p)}$.

Next, for deterministic $\eta \in \Omega_p$ and $\mathbf{q} \in V$, we define

$$(4.58) \qquad f_t(\eta) := \mathbb{E}\left[\prod_{i=1}^{M} \frac{1}{(2\eta_i - 1)!!} \left(\mathbf{q} \cdot \mathbf{v}_i(t)\right)^{2\eta_i} \,\middle|\, \left(\lambda_i(t) : i \in [\![1, M]\!], t \geqslant 0\right) \right],$$

where $n!! := n \cdot (n-2) \cdots 3 \cdot 1$ for odd n, and by convention $(-1)!! = 1$. In [24, Theorem 3.1] it is shown that f_t solves (4.57).

Remark 4.21. *In [24], Dyson Brownian motion is defined without the Ornstein-Uhlenbeck drift term in the SDE (4.2), and the SDEs for the eigenvalues and eigenvectors are stated in [24, Definition 2.2]. In the present case, with drift term, the SDEs for eigenvalue and eigenvector flows are given by*

$$d\lambda_i = \frac{dB_{ii}}{\sqrt{N}} + \frac{1}{N}\sum_{j:j\neq i}\frac{1}{\lambda_i - \lambda_j}dt - \frac{\lambda_i}{2}dt,$$

$$d\mathbf{v}_i = \frac{1}{\sqrt{N}}\sum_{j:j\neq i}\frac{dB_{ij}}{\lambda_i - \lambda_j}\mathbf{v}_j - \frac{1}{2N}\sum_{j:j\neq i}\frac{dt}{(\lambda_i - \lambda_j)^2}\mathbf{v}_i,$$

for $i = 1, 2, \ldots, M$, and with $B(t)$ a Brownian motion on the space of $M \times M$ real symmetric matrices with quadratic covariation $\langle B_{ij}, B_{kl}\rangle(t) = (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})t$. Thus, the SDEs for the eigenvectors are the same with or without the drift term. Therefore the arguments of [24, Section 3] apply verbatim in our setting as well.

Proof of Lemma 4.20. Suppose first that $H(0)$ has simple spectrum. Let f_t be the given by (4.58), which solves (4.57) as remarked above. Then, since the EMF (4.57) is a contraction on $L^\infty(\Omega_p)$, we obtain from the assumption of Lemma 4.20 that

$$\max_{\eta \in \Omega_p} |f_t(\eta)| \leqslant \max_{\eta \in \Omega_p} |f_0(\eta)| \leqslant B^{2p}.$$

Therefore, choosing $\eta = p\,\mathbf{e}_i$, we get

$$\mathbb{E}\big[(\mathbf{q} \cdot \mathbf{v}_i(t))^{2p}\big] = (2p-1)!! \, \mathbb{E}[f_t(\eta)] \leqslant (2p-1)!! B^{2p},$$

from which the claim follows. Finally, if $H(0)$ does not have simple spectrum, the same estimate holds by a simple approximation argument using the continuity of the eigenvectors as functions of the matrix. $\square$

4.3.2. *Stability of eigenvalues.* The following result shows that if the empirical spectral measure at $t = 0$ is close to the semicircle law, this remains true for $t > 0$. For its statement, recall that $s(t, z)$ denotes the Stieltjes transform of the empirical spectral measure of $H(t)$. We denote the Stieltjes transform of the semicircle law by m. It can be characterized as the unique holomorphic function $m : \mathbb{C}_+ \to \mathbb{C}_+$ such that $m^2 + mz + 1 = 0$ and $m(z) \sim 1/z$ as $|z| \to \infty$; see e.g. [16].

Lemma 4.22. *Suppose that $C^{-1}M \leqslant N \leqslant CM$. Fix $\varepsilon > 0$. If for some $B \leqslant N^{-\varepsilon}$ we have*

$$(4.59) \qquad |s(0; z) - m(z)| \prec B + \frac{1}{(N\eta)^{1/4}}$$

uniformly for $z = E + \mathrm{i}\eta$ with $\eta \geqslant N^{-1+\varepsilon}$, then for any $t \leqslant B$ we have

$$(4.60) \qquad |s(t; z) - m(z)| \prec B + \frac{1}{(N\eta)^{1/4}}.$$

uniformly for $z = E + \mathrm{i}\eta$ with $\eta \in [N^{-1+\varepsilon}, 1]$.

Proof. Define $s_{\text{fc},t}(z)$ as the unique solution $\mathbb{C}_+ \to \mathbb{C}_+$ of the self-consistent equation

$$(4.61) \qquad s_{\text{fc},t}(z) = \frac{1}{M} \sum_{i=1}^{M} \frac{1}{e^{-t/2}\lambda_i(0) - z - (1 - e^{-t})s_{\text{fc},t}(z)}.$$

Thus, $s_{\text{fc},t}(z)$ is the Stieltjes transform of the free convolution of the empirical eigenvalue distribution of $e^{-t/2}H(0)$ and the semicircle law rescaled by $(1 - e^{-t})^{1/2}$. We refer to [20] for the existence and uniqueness of $s_{\text{fc},t}(z)$ and relative properties on the free convolution with semicircle law.

As in (4.26), we find that $H(t) \overset{\text{d}}{=} e^{-t/2}H(0) + (1 - e^{-t})^{1/2}W$, where W is the standard Gaussian measure on $\mathcal{H}(V)$ with inner product (4.1). Under the assumptions of the lemma, [63, Corollary 7.11] implies that for $t \leqslant N^{-\varepsilon}$ we have

$$(4.62) \qquad |s(t; z) - s_{\text{fc},t}(z)| \prec \frac{1}{(N\eta)^{1/3}}$$

uniformly for $z = E + i\eta$ with $\eta \geqslant N^{-1+\varepsilon}$. (Note that in [63], the Stieltjes transform is denoted by m_V instead of s, and that $s_{\text{fc},t}$ is denoted $m_{\text{fc},t}$. Moreover, [63, Corollary 7.11] is stated for a diagonal matrix $H(0)$; however, since W is invariant under orthogonal transformations which diagonalize $H(0)$, the results of [63] trivially apply to any symmetric matrix $H(0)$.)

Set $\vartheta_t := 1 - e^{-t} \leqslant t$. Note that the Stieltjes transform of the empirical eigenvalue distribution of $e^{-t/2}H(0)$ is given by $e^{t/2}s(0, e^{t/2}z)$, and that (4.61) can be rephrased as

$$s_{\text{fc},t}(z) = e^{t/2}s(0, e^{t/2}(z + \vartheta_t s_{\text{fc},t}(z))).$$

For any $z = E + i\eta$ such that $\eta \geqslant N^{-1+\varepsilon}$, we have $\operatorname{Im} e^{t/2}(z + \vartheta_t s_{\text{fc},t}(z)) \geqslant \operatorname{Im} e^{t/2}z \geqslant N^{-1+\varepsilon}$, where we used that $\operatorname{Im} s_{\text{fc},t}(z) > 0$. From the assumption (4.59) we therefore get

$$(4.63) \qquad s_{\text{fc},t}(z) = e^{t/2}m(e^{t/2}(z + \vartheta_t s_{\text{fc},t}(z))) + O_{\prec}\left(B + \frac{1}{(N\eta)^{1/4}}\right).$$

Next, note that

$$(4.64) \qquad m(z) \ = \ \mathrm{e}^{t/2} m(\mathrm{e}^{t/2}(z + \vartheta_t m(z))).$$

(This may be interpreted as the fact that the semicircle law with variance t is a semigroup with respect to free convolution.) Moreover, from the definition of $m(z)$ it is easy to deduce the continuity estimate

$$(4.65) \qquad |m(z) - m(w)| \ \leqslant \ 2|z - w|^{1/2},$$

for any $z, w \in \mathbb{C}_+$.

By (4.65), and using that $t = \mathrm{O}(1)$, the difference between (4.63) and (4.64) is

$$
\begin{aligned}
|s_{\mathrm{fc},t}(z) - m(z)| \ &= \ \mathrm{e}^{t/2} \left| m(\mathrm{e}^{t/2}(z + \vartheta_t s_{\mathrm{fc},t}(z))) - m(\mathrm{e}^{t/2}(z + \vartheta_t m(z))) \right| + \mathrm{O}_{\prec}\left(B + \frac{1}{(N\eta)^{1/4}}\right) \\
&\leqslant \ \mathrm{O}(t^{1/2}) |s_{\mathrm{fc},t}(z) - m(z)|^{1/2} + \mathrm{O}_{\prec}\left(B + \frac{1}{(N\eta)^{1/4}}\right) \\
&\leqslant \ \max\left\{ \mathrm{O}(t^{1/2}) |s_{\mathrm{fc},t}(z) - m(z)|^{1/2}, \mathrm{O}_{\prec}\left(B + \frac{1}{(N\eta)^{1/4}}\right) \right\}.
\end{aligned}
$$

Therefore either $|s_{\mathrm{fc},t}(z) - m(z)| = \mathrm{O}(t)$ or $|s_{\mathrm{fc},t}(z) - m(z)| \prec B + (N\eta)^{-1/4}$, and we get

$$(4.66) \qquad |s_{\mathrm{fc},t}(z) - m(z)| \ \prec \ B + \frac{1}{(N\eta)^{1/4}} + t \ \prec \ B + \frac{1}{(N\eta)^{1/4}},$$

where we used $t \leqslant B$. Combining (4.62) and (4.66) and using $\eta \leqslant 1$, the claim (4.60) follows. $\qquad \square$

4.4. **Proof of Propositions 4.6–4.7.** With the preparations provided by Sections 4.2–4.3, and using results of [19, 63, 52], we now complete the proofs of Propositions 4.6–4.7. First, recall that $\alpha > 0$ is fixed, and that we always assume $D \geqslant N^{\alpha}$. We also use the notation $z = E + \mathrm{i}\eta$ for the real and imaginary parts of the spectral parameter $z \in \mathbb{C}_+$.

Throughout this section, $H(t)$ denotes the constrained DBM from Definition 4.2 with $H(0)$ given by (4.4). We use the notations of Section 4.3 applied to the constrained DBM. In particular,

$$M := N - 1$$

is the dimension of the space $V := \mathbf{e}^{\perp}$.

4.4.1. *A priori estimates on eigenvalues and eigenvectors.* We begin by collecting some results on the eigenvalues and eigenvectors of $H(t)|_{\mathbf{e}^{\perp}}$ required for the proofs of Propositions 4.6–4.7.

For any $H \in \mathcal{M}$, we denote the eigenvalues of $H|_{\mathbf{e}^{\perp}}$ by $\lambda_1(H) \geqslant \cdots \geqslant \lambda_M(H)$, and the corresponding normalized eigenvectors by $\mathbf{v}_1(H), \ldots, \mathbf{v}_M(H)$. The components of the eigenvectors in the standard basis on $\mathbb{R}^N$ are denoted $v_k(H; i) := \mathbf{e}_i \cdot \mathbf{v}_k(H)$, $i \in [\![1, N]\!]$, $k \in [\![1, M]\!]$. Moreover, for $H \in \mathcal{M}$, we denote by $G_{ij}(H; z)$ the entries of the Green's function of H restricted to $\mathbf{e}^{\perp}$ in the standard basis of $\mathbb{R}^N$, and by $s(H; z)$ the Stieltjes transform of the empirical spectral measure. Explicitly,

$$(4.67) \qquad G_{ij}(H; z) := \sum_{k=1}^{M} \frac{v_k(H; i) v_k(H; j)}{\lambda_k(H) - z},$$

$$(4.68) \qquad s(H; z) := \frac{1}{M} \operatorname{Tr} G(H; z) = \frac{1}{M} \sum_{k=1}^{M} \frac{1}{\lambda_k(H) - z}.$$

Finally, we set

$$(4.69) \qquad \Gamma(H) \equiv \Gamma(H; z) := \max_{i,j} |G_{ij}(H; z)| \vee 1.$$

We also recall the definition of the typical location γ_i of the i-th eigenvalue from (1.6).

The following proposition summarizes the input we need from the local semicircle law of [19]. The local semicircle law, as proved in [19], only applies for $t = 0$, and the extension to $t > 0$ is provided by the results of Section 4.3.

Proposition 4.23. *For any $z \in \mathbb{C}_+$, $i \in [\![1, N]\!]$, $k \in [\![1, M]\!]$, and $0 \leqslant t \leqslant D^{-1/4}$, we have*

$$(4.70) \qquad |v_k(H(t); i)| \prec N^{-1/2}, \qquad \Gamma(H(t); z) \prec 1 + \frac{1}{N\eta}.$$

Moreover, for any fixed $\kappa > 0$ and any $i \in [\![\kappa N, (1 - \kappa)N]\!]$, we also have

$$(4.71) \qquad |\lambda_i(H(t)) - \gamma_i| \prec D^{-1/4}.$$

Proof. First, as special cases of [19, Theorem 1.1 and Corollary 1.2], for any $z = E + i\eta$ with $E \in \mathbb{R}$ and $\eta \geqslant N^{-1+\varepsilon}$, for arbitrary $\varepsilon > 0$, we have

$$(4.72) \qquad |s(H(0); z) - m(z)| \prec \frac{1}{D^{1/4}} + \frac{1}{(N\eta)^{1/4}}, \qquad |v_k(H(0); i)| \prec N^{-1/2}.$$

(Note that the local semicircle law from [19] also includes the trivial eigenvalue at 0; it is easy to see that its contribution to s is negligible compared to the error bounds in (4.72).)

Next, we extend these bounds from $t = 0$ to $t > 0$. For $i \in [\![1, N]\!]$ define $\hat{\mathbf{e}}_i = \mathbf{e}_i - (\mathbf{e}_i \cdot \mathbf{e})\mathbf{e} \in \mathbf{e}^\perp$. Since $v_k(H(t); i) = \hat{\mathbf{e}}_i \cdot \mathbf{v}_k(H(t))$, from (4.72) and Lemma 4.20, applied to the constrained DBM with $\mathbf{q} = \hat{\mathbf{e}}_i$, we find $|v_k(H(t); i)| \prec N^{-1/2}$, for any $t > 0$. Similarly, for $t \leqslant D^{-1/4}$, the extension of the bound on the Stieltjes transform follows immediately from Lemma 4.22 with $B = D^{-1/4}$. Summarizing, for any $\eta \geqslant N^{-1+\varepsilon}$ and $0 \leqslant t \leqslant D^{-1/4}$, we have

$$(4.73) \qquad |s(H(t); z) - m(z)| \prec \frac{1}{D^{1/4}} + \frac{1}{(N\eta)^{1/4}}, \qquad |v_k(H(t); i)| \prec N^{-1/2}.$$

This proves the first estimate of (4.70).

In order to prove the second estimate of (4.70), we use a dyadic decomposition (see e.g. [45, (8.2)]) to obtain, for any matrix $H \in \mathcal{M}$,

$$|G_{ij}(z)| \leqslant \sum_{k=1}^{M} \frac{|v_k(i)v_k(j)|}{|\lambda_k - E + \mathrm{i}\eta|} \leqslant 4N \max_{k,l} |v_k(l)|^2 \left(1 + \sum_{n=0}^{\lceil \log_2 \eta^{-1} \rceil} \mathrm{Im}\, s(E + \mathrm{i}2^n \eta)\right).$$

We apply this estimate to the matrix $H(t)$. By (4.73), we have $\max_{k,l} |v_k(l)|^2 \prec 1/N$. Moreover, since $\eta\, \mathrm{Im}\, s(E + \mathrm{i}\eta)$ is increasing in η (as may be easily seen from the right-hand side of (4.68)), and since $|m| \leqslant 1$, the first bound in (4.73) implies $\mathrm{Im}\, s(z) \prec 1 + 1/(N\eta)$ for any $\eta > 0$, and thus $\mathrm{Im}\, s(E + \mathrm{i}2^n \eta) \prec 1 + 2^{-n}/N\eta$. For $\eta \geqslant 1/N^{\mathrm{O}(1)}$ we then have $\log \eta^{-1} \prec 1$ and obtain $\Gamma(z) \prec 1$ as desired. For arbitrary $\eta > 0$ the claim then follows by [19, Lemma 2.1]. (In fact, we shall only need (4.70) with $\eta \geqslant 1/N^{\mathrm{O}(1)}$.)

Finally, we deduce (4.71) from the bound on the Stieltjes transform in (4.73). We abbreviate $\lambda_k \equiv \lambda_k(H(t))$, and denote by

$$\rho_{sc}(I) := \int_I \rho_{sc}(x)\, \mathrm{d}x, \qquad \nu(I) := \frac{1}{M} \sum_{k=1}^{M} \mathbf{1}(\lambda_k \in I)$$

the semicircle and empirical spectral measures, respectively, applied to an interval I. Then, following a standard application of the Helffer-Sjöstrand functional calculus along the lines of [39, Section 8.1], we find from (4.73) and $D \leqslant N$ that for any interval $I \subseteq [-3, 3]$ we have

(4.74)
$$|\nu(I) - \rho_{sc}(I)| \prec \frac{1}{D^{1/4}} + \frac{1}{N^{1/4}} \prec \frac{1}{D^{1/4}}.$$

(We note that previously (4.74) for $t = 0$ was given in [19, Corollary 1.3].) Using (4.74), we may estimate $\lambda_i - \gamma_i$ as follows. By (4.74) applied to $I = [-3, 3]$, we find that there are at most $\mathrm{O}_\prec(ND^{-1/4})$ eigenvalues outside $[-3, 3]$. Defining $f(E) :=$

$\rho_{sc}([E, \infty))$, we therefore find from (1.6) and (4.74) that

$$f(\gamma_i) \;=\; \frac{i}{N} \;=\; \nu([\lambda_i, \infty)) + O\Big(\frac{1}{N}\Big) \;=\; \nu([\lambda_i, 3)) + O_{\prec}\Big(\frac{1}{D^{1/4}}\Big)$$
$$=\; \rho_{sc}([\lambda_i, 3)) + O_{\prec}\Big(\frac{1}{D^{1/4}}\Big) \;=\; f(\lambda_i) + O_{\prec}\Big(\frac{1}{D^{1/4}}\Big)\,.$$

Since $i \in [\![\kappa N, (1 - \kappa N)]\!]$, we have $|f'| \geqslant c > 0$ in a neighbourhood of γ_i, and we therefore get (4.71). This concludes the proof. $\qquad\square$

The next result shows that the suprema in (4.18) do not essentially change the size of Γ.

Corollary 4.24. *Fix $n \in \mathbb{N}$. For any $z \in \mathbb{C}_+$ and $0 \leqslant t \leqslant D^{-1/4}$, we have*

$$(4.75) \qquad \sup_{\theta \in [0,1]^n} \sup_{X \in \mathcal{X}^n} \Gamma\big(H(t) + (d-1)^{-1/2}\,\theta \cdot X; z\big) \;\prec\; 1 + \frac{1}{N\eta}\,.$$

Moreover, for any $i \in [\![1, N]\!]$ and $k \in [\![1, M]\!]$, we have

$$(4.76) \qquad \sup_{\theta \in [0,1]^n} \sup_{X \in \mathcal{X}^n} \big|v_k\big(H(t) + (d-1)^{-1/2}\,\theta \cdot X; i\big)\big| \;\prec\; N^{-1/2}\,.$$

Proof. We abbreviate $H \equiv H(t)$. Without loss of generality, by an argument analogous to [19, Lemma 2.1], we may assume that $\eta \geqslant 1/N$. Hence, by (4.70), we have $\Gamma(H; z) \prec 1$. It therefore suffices to show that if $\Gamma(H; z) \leqslant (d-1)^{1/2}/(16n)$ then for any $\theta \in [0, 1]^n$ and $X \in \mathcal{X}^n$ we have

$$(4.77) \qquad \Gamma\big(H + (d-1)^{-1/2}\,\theta \cdot X; z\big) \;\leqslant\; 2\Gamma(H; z)\,.$$

To show (4.77), we use the resolvent identity to obtain (omitting the argument z for brevity)

$$\left|G_{ij}\big(H + (d-1)^{-1/2}\theta \cdot X\big)\right|$$

$$= \left|G_{ij}(H) - (d-1)^{-1/2}\Big(G(H)(\theta \cdot X)G\big(H + (d-1)^{-1/2}\theta \cdot X\big)\Big)_{ij}\right|$$

$$\leqslant \Gamma(H) + 8n(d-1)^{-1/2}\Gamma(H)\Gamma\big(H + (d-1)^{-1/2}\theta \cdot X\big)$$

$$\leqslant \Gamma(H) + \Gamma\big(H + (d-1)^{-1/2}\theta \cdot X\big)/2.$$

Taking the maximum over i and j yields (4.77). Finally, (4.76) follows from (4.75), as in the proof of [19, Corollary 1.2]. $\square$

Note that since $G_{ij}(H;\bar{z}) = \overline{G_{ij}(H;z)}$, the estimates (4.70) and (4.75) for Γ also hold with $\eta < 0$ if η is replaced by $|\eta|$ on the right-hand sides. We shall use this tacitly in the following.

4.4.2. *Proof of Proposition 4.6: eigenvalue correlation functions.* We now prove that the locally averaged local correlation functions of the matrix $H(0)|_{\mathbf{e}^{\perp}}$ converge to those of $H(t)|_{\mathbf{e}^{\perp}}$ for times $t \leqslant N^{-1-\delta}D^{1/2}$. The main ingredient of the proof is the following lemma comparing functions of Green's functions with spectral parameter η slightly smaller than $1/N$. Its proof follows easily from Proposition 4.8 and Lemma 4.23. For random matrices with independent entries, analogous results were previously proved by the Green's function comparison theorem [45], and by direct analysis of the evolution of the matrix entries under Dyson Brownian motion [24]. We also remark that, in [86], eigenvalues are compared directly without involving the Green's function.

Lemma 4.25. *Fix $n \in \mathbb{N}$, and let $\kappa, \gamma, \delta > 0$ be sufficiently small. Then the following holds for any $\eta \in [N^{-1-\gamma}, N^{-1}]$, any sequence of positive integers $k_1, k_2, \ldots, k_n$, any set of complex parameters $z_j^m = E_j^m \pm \mathrm{i}\eta$, where $j \in [\![1, k_m]\!]$, $m \in [\![1, n]\!]$, $|E_j^m| \leqslant 2 - \kappa$,*

and the $\pm$ signs are arbitrary. Let $\phi \in C^\infty(\mathbb{R}^n)$ be a test function such that for any multi-index $m = (m_1, \cdots, m_n)$ with $1 \leqslant |m| \leqslant 4$ and for any $\omega > 0$ sufficiently small,

(4.78)
$$\max \left\{ |\partial^m \phi(x)| : |x| \leqslant N^\omega \right\} \leqslant N^{O(\omega)},$$

(4.79)
$$\max \left\{ |\partial^m \phi(x)| : |x| \leqslant N^2 \right\} \leqslant N^{O(1)}.$$

Then, with the notations $G_1(z) := G(H(0); z)$ and $G_2(z) := G(H(t); z)$, for any $t \leqslant D^{1/2} N^{-1-\delta}$, we have

(4.80)
$$\left| \mathbb{E}\phi \left(N^{-k_1} \operatorname{Tr} \left(\prod_{j=1}^{k_1} G_1(z_j^1) \right), \ldots, N^{-k_n} \operatorname{Tr} \left(\prod_{j=1}^{k_n} G_1(z_j^n) \right) \right) - \mathbb{E}\phi(G_1 \to G_2) \right|$$
$$= O(N^{-\delta/2 + O(\gamma)}).$$

Here, $\phi(G_1 \to G_2)$ is the expression obtained from the one to its left by replacing G_1 with G_2. The implicit constants depend on n, $k_1, \ldots, k_n$, $m_1, \ldots, m_n$, and the constants in (4.78)–(4.79).

Proof. For simplicity of notation, we show (4.80) only for $n = 1$ and $k_1 = 1$; the general case is analogous. We then write z instead of z_1^1. To show the claim, it then suffices to show that

(4.81)
$$\left| \mathbb{E}\phi \left(N^{-1} \operatorname{Tr} G(H(t); z) \right) - \mathbb{E}\phi \left(N^{-1} \operatorname{Tr} G(H(0); z) \right) \right| = O\left(t D^{-1/2} N^{1+\delta/2} N^{O(\gamma)} \right).$$

Set $F(H) := \phi(N^{-1} \operatorname{Tr} G(H; z))$. We claim that if r and n are fixed (arbitrarily, independently of N), and if $t \leqslant D^{-1/4}$, for any sufficiently large N (depending on r, n, δ), we have

(4.82)
$$\sup_{0 \leqslant s \leqslant t} \|\partial^n F\|_{r,s} \leqslant N^{\delta/4 + O(\gamma)}.$$

Given (4.82), Proposition 4.8 with $\varepsilon = \delta/4$ yields (4.81).

Thus, it only remains to show (4.82). Recall that the derivative of the Green's function in the direction of a matrix $X \in \mathcal{X}$ is given by $\partial_X G = -GXG$ (using that elements in $\mathcal{X}$ act on $\mathbf{e}^\perp$). Therefore, by the Leibniz rule, for any $X_1, \ldots, X_n \in \mathcal{X}$ and any $H \in \mathcal{M}$, we have

$$\partial_{X_1} \cdots \partial_{X_n} G = (-1)^n \sum_{\sigma \in S_n} GX_{\sigma(1)} G \cdots GX_{\sigma(n)} G \,,$$

where S_n is the set of permutations of n elements, and we omit the dependence on H on both sides in our notation. Since (with respect to the standard basis of $\mathbb{R}^N$) each $X \in \mathcal{X}$ has at most 8 nonvanishing entries, and since these are in $\{\pm 1\}$, by definition of Γ it follows that

$$\left| N^{-1} \operatorname{Tr} \partial_{X_1} \cdots \partial_{X_n} G \right| \leqslant N^{-1} \sum_{i=1}^N n! \max_{\sigma \in S_n} \left| \left(GX_{\sigma(1)} G \cdots GX_{\sigma(n)} G \right)_{ii} \right| \leqslant 8^n n! \Gamma^{n+1} \,.$$

From this and the chain rule, we obtain that there exist constants C_n such that

$$(4.83) \qquad \left| \partial_{X_1} \cdots \partial_{X_n} \phi(N^{-1} \operatorname{Tr} G) \right| \leqslant C_n \Gamma^{2n} \max_{0 \leqslant m \leqslant n} \left| \phi^{(m)} \right| \,.$$

By Corollary 4.24 and since $|\eta| \geqslant N^{-1-\gamma}$, we have $\sup_{\theta \in [0,1]^n} \sup_{X \in \mathcal{X}^n} \Gamma(H(s) + (d - 1)^{-1/2} \theta \cdot X) \prec N^\gamma$, for any $0 \leqslant s \leqslant t$. For $n \leqslant 4$, by assumption (4.78) and (4.83) therefore

$$(4.84) \quad \sup_{\theta \in [0,1]^n} \sup_{X \in \mathcal{X}^n} \left| \partial_{X_1} \cdots \partial_{X_n} \phi\big(N^{-1} \operatorname{Tr} G\big(H(s) + (d - 1)^{-1/2} \theta \cdot X\big)\big) \right| \prec N^{O(\gamma)} \,.$$

On the complement of the high-probability event of $\prec$ in (4.84), we use the trivial bound $\Gamma \leqslant \eta^{-1} \leqslant N^{1+\gamma}$ and (4.79). We obtain

(4.85)

$$\sup_{\theta \in [0,1]^n} \sup_{X \in \mathcal{X}^n} \left| \partial_{X_1} \cdots \partial_{X_n} \phi\big(N^{-1} \operatorname{Tr} G\big(H(s) + (d - 1)^{-1/2} \theta \cdot X\big)\big) \right| \leqslant C_n \eta^{-2n} N^{O(1)} \leqslant N^{O(1)} \,,$$

for any $0 \leqslant s \leqslant t$. By combining the estimates (4.84)–(4.85), for any constant $r = \mathrm{O}(1)$, we have

$$(4.86) \qquad \|\partial^n F\|_{r,s} \ \leqslant \ N^{1/\zeta + \mathrm{O}(\gamma)} + N^{-\zeta/r + \mathrm{O}(1)} \ \leqslant \ N^{\delta/4 + \mathrm{O}(\gamma)},$$

where ζ is as in Definition 4.19 and chosen sufficiently large, depending on r. This concludes the proof. $\qquad\square$

The following lemma is essentially [45, Theorem 6.4]. It transforms the statement about the Green's function of Lemma 4.25 to a statement about the local correlation functions.

Lemma 4.26. *Consider two random matrix ensembles H_1 and H_2 with Green's functions $G_1(z)$ and $G_2(z)$. Suppose that, for all ϕ and parameters as in the statement of Lemma 4.25, the estimate (4.80) holds. Then the local bulk eigenvalue correlation functions of H_1 and H_2 coincide.*

Proof of Proposition 4.6: correlation functions. The proof follows directly by combining Lemmas 4.25–4.26, with δ given as in the assumption of Proposition 4.6. $\quad\square$

4.4.3. *Proof of Proposition 4.6: eigenvalue gap statistics.* To prove that the eigenvalue gap statistics are stable for short times, we require a weak level repulsion estimate. Such an estimate was derived in [52, Theorem 4.1] for sparse matrices with independent entries, using a level repulsion estimate for $t \geqslant N^{-1+c}$ established in [63]. Here we adapt the proof of [52, Theorem 4.1] to random regular graphs. The nontrivial dependence is dealt with by Proposition 4.8.

If $\lambda_i(H)$ is a simple eigenvalue of $H|_{\mathbf{e}^\perp}$, we define

$$(4.87) \qquad Q_i(H) \ := \ \frac{1}{N^2} \sum_{j:j\neq i, j \leqslant M} \frac{1}{(\lambda_j(H) - \lambda_i(H))^2},$$

and extend this definition by $Q_i(H) := \infty$ if $\lambda_i(H)$ is not a simple eigenvalue. This quantity plays an important role in [86], where it is observed that it captures the

singularities of the derivatives of $\lambda_i(H)$. In [52], it is found that Q_i is stable under DBM and can thus be used to show weak level repulsion from such an estimate for larger times (when a Gaussian component is present).

Proposition 4.27 (Level repulsion). *Fix $\kappa > 0$. Then for any sufficiently small $\tau > 0$, any $i \in [\![\kappa N, (1 - \kappa)N]\!]$, and any $s \geqslant 0$, we have*

$$(4.88) \qquad \mathbb{P}\big(Q_i(H(s)) \geqslant N^{2\tau}\big) \;=\; \mathrm{O}(N^{-\tau/2}) \,.$$

In particular,

$$(4.89) \qquad \mathbb{P}\big(\lambda_i(H(s)) - \lambda_{i+1}(H(s)) \leqslant N^{-1-\tau}\big) \;=\; \mathrm{O}(N^{-\tau/2}) \,.$$

Proof. The proof is analogous to that of [52, Theorem 4.1], with $H|_{\mathbf{e}^\perp}$ instead of H. We here focus on the differences. These result from the replacement of [52, Lemma 4.3] by Proposition 4.8, which takes into account the nontrivial correlation structure of the random regular graph. As in [52], if $\lambda_i(H)$ is a simple eigenvalue of $H|_{\mathbf{e}^\perp}$, we define the matrix

$$R_i(H) \;:=\; \sum_{j:j\neq i, j\leqslant M} \frac{1}{\lambda_i(H) - \lambda_j(H)} \mathbf{v}_j(H)\mathbf{v}_j(H)^* \;=\; \frac{1}{2\pi i} \oint_{|z-\lambda_i(H)|=\omega} \frac{G(H;z)}{\lambda_i(H) - z} \, \mathrm{d}z \,,$$

where ω is chosen such that the contour $|z - \lambda_i(H)| = \omega$ encloses only $\lambda_i(H)$. Then we have

$$Q_i(H) \;=\; \frac{1}{N^2} \operatorname{Tr}(R_i(H)^2) \,.$$

Given $\tau > 0$, define a cutoff function χ satisfying the following two properties: (1) χ is smooth, and the first four derivatives are bounded, i.e. $|\chi^{(k)}(x)| = \mathrm{O}(1)$, for $k = 1, 2, 3, 4$; (2) On the interval $[0, N^{2\tau}]$, $|\chi(x) - x| \leqslant 1$, and for $x \geqslant N^{2\tau}$, $\chi(x) = N^{2\tau}$. Then $\chi \circ Q_i$ extends to a smooth function on the space of symmetric matrices.

The proof of (4.88) consists of three steps. The first step is the estimate

$$(4.90) \qquad \mathbb{E}[\chi(Q_i(H(s)))] = \mathrm{O}(N^{3\tau/2}),$$

for $s \geqslant t := N^{-1+c}$. This estimate follows from [63, Theorem 3.6], whose assumptions are satisfied with high probability for the random d-regular graph by Proposition 4.23. In particular, independence of the entries of H is not used.

In the second step, we derive the comparison estimate

$$(4.91) \qquad \left| \mathbb{E}[\chi(Q_i(H(t)))] - \mathbb{E}[\chi(Q_i(H(s)))] \right| \leqslant 1,$$

for $s \in [0, t]$. Instead of using [52, Lemma 4.3], which requires that the entries of the random matrix $H(s)$ are independent, we use Proposition 4.8, which takes into account the nontrivial correlation structure of the random regular graph. By Proposition 4.8 with $F(H) := \chi(Q_i(H))$, it suffices to bound

(4.92)

$$\|\partial^n F\|_{r,s} = \mathbb{E} \left[\sup_{\theta \in [0,1]^n} \sup_{X \in \mathcal{X}^n} \left| \partial_{X_1} \partial_{X_2} \cdots \partial_{X_n} F\big(H(s) + (d-1)^{-1/2}\,\theta \cdot X\big) \right|^r \right]^{1/r},$$

for any (large) fixed integer r and $n = 1, 2, 3, 4$. To this end, the computation of the proof of [52, Proposition 4.6] applies, by simply replacing the derivatives $\partial_{ab}^{(n)}$ by $\partial_{X_1} \cdots \partial_{X_n}$ with $X_l \in \mathcal{X}$. Here the formulas [52, (4.16)–(4.18)] remain valid after replacing V by the X_l appropriately, and similarly the formula below [52, (4.18)] remains valid after replacing V_{ij} by $\mathbf{v}_i^*(H) X_l \mathbf{v}_j(H)$. Moreover, an analogous formula holds for $n = 4$; see e.g. [79, p.8]. The same formulas are valid with H replaced by $H + (d-1)^{-1/2}\,\theta \cdot X$. Since the X_l have only 8 nonvanishing entries (in the standard basis on $\mathbb{R}^N$), and these are equal to ± 1, Corollary 4.24 then implies

$$\sup_{\theta \in [0,1]^n} \sup_{X \in \mathcal{X}^n} \left| \mathbf{v}_i^*\big(H(s) + (d-1)^{-1/2}\,\theta \cdot X\big) X_l\, \mathbf{v}_j\big(H(s) + (d-1)^{-1/2}\,\theta \cdot X\big) \right| \prec N^{-1}$$

for any $s \in [0, t]$. As in the proof of [52, Proposition 4.6], we therefore get

$$\sup_{\theta \in [0,1]^n} \sup_{X \in \mathcal{X}^n} \left| \partial_{X_1} \partial_{X_2} \cdots \partial_{X_n} F\big(H(s) + (d-1)^{-1/2}\, \theta \cdot X\big) \right| \prec N^{(n+2)\tau}\, .$$

From this, bounding (4.92) as in (4.86), we obtain

$$(4.93) \qquad \qquad \|\partial^n F\|_{r,s} \leqslant N^{c+(n+2)\tau}\, .$$

for arbitrarily small $c > 0$ and N large enough. Then (4.91) follows from Proposition 4.19 since $\mathrm{O}(tD^{-1/2}N)N^{c+6\tau} \leqslant \mathrm{O}(N^{-\alpha/2+2c+6\tau}) \leqslant 1$ for $t \leqslant N^{-1+c}$ and $D \geqslant N^{\alpha}$, by chooosing c and τ sufficiently small.

In the last step, we combine (4.90) and (4.91), and thus obtain

$$\mathbb{E}[\chi(Q_i(H(s)))] \;=\; \mathrm{O}(N^{3\tau/2})\, ,$$

for any $s \geqslant 0$. Then (4.88) follows easily by Markov's inequality and the definition of χ. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proof of Proposition 4.6: gap statistics. Throughout the proof, we use the abbreviation $\lambda_i(t) \equiv \lambda_i(H(t))$. Fix $\kappa > 0$, $\delta > 0$, and $t \leqslant N^{-1-\delta}D^{1/2}$. Since $\rho_{sc}(\gamma_i)$ is bounded above and below for $i \in [\![\kappa N, (1-\kappa)N]\!]$, it suffices to prove (4.14) with $\rho_{sc}(\gamma_i)$ replaced by 1. Moreover, for any $n \in \mathbb{N}$ and $\phi \in C^\infty(\mathbb{R}^n)$ with bounded first four derivatives, it suffices to show the stronger claim

$$(4.94) \qquad \mathbb{E}\phi\big(N\lambda_i(0), \ldots, N\lambda_{i+n}(0)\big) \;=\; \mathbb{E}\phi\big(N\lambda_i(t), \ldots, N\lambda_{i+n}(t)\big) + \o(1)$$

as $N \to \infty$, uniformly in $i \in [\![\kappa N, (1-\kappa)N]\!]$. For simplicity of notation, we only prove (4.94) for $n = 1$; the general case is analogous and we comment on the differences at the end of the proof. Thus, for any $i \in [\![\kappa N, (1-\kappa)N]\!]$ and $\phi \in C^\infty(\mathbb{R})$ with bounded first four derivatives, we show

$$(4.95) \qquad \qquad \mathbb{E}[\phi(N\lambda_i(0))] - \mathbb{E}[\phi(N\lambda_i(t))] \;=\; \o(1)\, .$$

Given a small constant $\tau > 0$, we choose a cutoff function ρ such that $\rho(x) = 1$ for $x \leqslant N^{2\tau}$ and $\rho(x) = 0$ for $x \geqslant 2N^{2\tau}$. Using (4.88), we can first remove a bad event on which Q_i is large:

$$\left| \mathbb{E}[\phi(N\lambda_i(0))] - \mathbb{E}[\phi(N\lambda_i(t))] \right|$$

$$\leqslant \left| \mathbb{E}[\phi(N\lambda_i(0))\rho(Q_i(H(0)))] - \mathbb{E}[\phi(N\lambda_i(t))\rho(Q_i(H(t)))] \right|$$

$$+ \|\phi\|_\infty \left(\mathbb{P}(Q_i(H(0)) \geqslant N^{2\tau}) + \mathbb{P}(Q_i(H(t)) \geqslant N^{2\tau}) \right)$$

$$\leqslant \left| \mathbb{E}[\phi(N\lambda_i(0))\rho(Q_i(H(0)))] - \mathbb{E}[\phi(N\lambda_i(t))\rho(Q_i(H(t)))] \right| + O\left(\frac{\|\phi\|_\infty}{N^{\tau/2}} \right).$$

To estimate the right-hand side, we apply Proposition 4.8 with $F(H) := \phi(N\lambda_i(H))\rho(Q_i(H))$. By an argument analogous to that used to obtain (4.93), for any r and $n = 1, 2, 3, 4$, we find the bound

$$(4.96) \qquad \qquad \|\partial^n F\|_{r,s} \leqslant N^{c+O(\tau)}$$

for arbitrarily small $c > 0$ (and N sufficiently large). More precisely, by the product rule, the derivatives act either on $\phi(N\lambda_i)$ or $\rho \circ Q_i$. In the bound of any of these derivatives, by definition of ρ, we can assume that $Q_i \leqslant 2N^{2\tau}$. Then the derivatives of $\rho \circ Q_i$ are bounded exactly as in the proof of Proposition 4.27. For the derivatives of $\phi(N\lambda_i)$, by the chain rule and since ϕ is smooth, it suffices to bound the derivatives of the eigenvalues λ_i. This is again done similarly to the bounds on the derivatives of Q_i. Indeed, the derivatives of the eigenvalues can be expressed in terms of the eigenvalues and eigenvectors as done in [52, (4.16)–(4.18)] (and with [79, p.8] for $n = 4$). The latter expressions are bounded using the delocalization of eigenvectors (4.70), and using that

$$\sum_{j:j\neq i} \frac{1}{|\lambda_i(s) - \lambda_j(s)|} \prec NQ_i^{1/2}(H(s)), \qquad \sum_{j:j\neq i} \frac{1}{|\lambda_i(s) - \lambda_j(s)|^k} \leqslant N^k Q_i^{k/2}(H(s)),$$

as in [52, (4.11)–(4.12)].

As a consequence of Proposition 4.8 and (4.96), with $t \leqslant N^{-1-\delta}D^{1/2}$, we finally obtain

$$|\mathbb{E}[\phi(N\lambda_i(0))\rho(Q_i(H(0)))] - \mathbb{E}[\phi(N\lambda_i(t))\rho(Q_i(H(t)))]| = \mathrm{O}(N^{c+\mathrm{O}(\tau)-\delta}),$$

and (4.95) then follows by taking c and τ small enough that $c + \mathrm{O}(\tau) < \delta$.

In the general case of a test function $\phi(N\lambda_i, \ldots, N\lambda_{i+n})$, we use the product of cutoff functions $(\rho \circ Q_i) \cdots (\rho \circ Q_{i+n})$ instead of $\rho \circ Q_i$, and proceed otherwise analogously. $\qquad\square$

4.4.4. *Proof of Proposition 4.7.*

Proof of Propositions 4.7. Given the estimates (4.70)–(4.71), the same argument as in [52, Section 3] applies. $\qquad\square$

APPENDIX A. COMBINATORIAL ESTIMATES FOR RANDOM REGULAR GRAPHS

A.1. **Proof of Proposition 2.1.**

Proof of (2.6). For $\omega = 1$, a proof of the statement is given in [65, Lemma 2.1] or [21, Lemma 7], for example. The more general statement follows from the same proof. More precisely, in [65, (2.4)], it is shown that for any $i \in [\![N]\!]$, the excess X_i in $\mathcal{B}_R(i, \mathcal{G})$ is stochastically dominated by a binomial random variable with $n = d(d-1)^R$ trials and success probability $p = d(d-1)^{R-1}/N$. It follows that

$$\mathbb{P}(X_i > \omega) = O\left(\binom{d(d-1)^R}{\omega+1}\left(\frac{d(d-1)^{R-1}}{N}\right)^{\omega+1}\right) = \mathrm{O}(N^{-\omega-1}(d-1)^{2R(\omega+1)}) = \mathrm{O}(N^{-\omega-1+2\kappa(\omega+1)}).$$

By a union bound, and using $\kappa < \delta/(2\omega + 2)$, therefore

$$\mathbb{P}(X_i > \omega \text{ for some } i \in [\![N]\!]) = \mathrm{O}(N^{-\omega+2\kappa(\omega+1)}) = \o(N^{-\omega+\delta}),$$

as claimed. $\qquad\square$

Proof of (2.7). The claim follows from [72, Theorem 4], for example. Indeed, if $\mathcal{B}_R(i, \mathcal{G})$ is not a tree, then some edge in $\mathcal{B}_R(i, \mathcal{G})$ must lie on a cycle of length at most $k = 2R$, and any edge that lies on such a cycle is in $\mathcal{B}_R(j, \mathcal{G})$ for at most $2(d-1)^R$ vertices $j \in [\![N]\!]$. Thus

$$(A.1) \qquad |\{i \in [\![N]\!] : \mathcal{B}_R(i, \mathcal{G}) \text{ is not a tree}\}| \leqslant 2(d-1)^R X = 2N^\kappa X$$

where X is the number of edges in $\mathcal{G}$ which lie on cycles of length at most k. With $k = 2R$ and $A \geqslant 2$ in [72, Theorem 4], we obtain

$$(A.2) \qquad \mathbb{P}(X = M) \leqslant (e^{5(A-1)} A^{-5A})^{(d-1)^k} \leqslant e^{-c(d-1)^k} = e^{-cN^{2\kappa}}$$

if $M = 20Ak(d-1)^k$, where c is some universal constant. Let $M_0 = 40k(d-1)^k \leqslant 80R(d-1)^{2R} \leqslant 80RN^{2\kappa}$. By a union bound, then

$$(A.3) \qquad \mathbb{P}(X \geqslant M_0) \leqslant Ne^{-cN^{2\kappa}} \leqslant e^{-cN^{2\kappa}/2}.$$

Thus, with probability $1 - e^{-cN^{2\kappa}/2}$, and using $\kappa < \delta/(2\omega+2) \leqslant \delta/4$, $R = \lfloor \kappa \log_{d-1} N \rfloor \ll N^\kappa$, we have

$$(A.4) \quad |\{i \in [\![N]\!] : \mathcal{B}_R(i, \mathcal{G}) \text{ is not a tree}\}| \leqslant 2N^\kappa X \leqslant 2N^\kappa M_0 \leqslant 160RN^{3\kappa} \leqslant N^\delta,$$

which is better than claimed. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

A.2. **Proof of Proposition 2.2.**

Proof of (2.8). We fix vertices i, j and an integer k. Given a graph $\mathcal{G}$, we denote by $t_k(\mathcal{G})$ the total number of non-backtracking paths from i to j of length less than $\text{dist}_\mathcal{G}(i, j) + k$. We modify the graph $\mathcal{G}$ in the three steps such that, in each step, t_k does not decrease, and the excess remains the same. Then it suffices to prove (2.8) for the final graph.

Step 1. Given an edge $e = \{x, y\} \in \mathcal{G}$ that is not a self-loop and not on a geodesic from i to j, we shrink the edge e to a point (remove e and identify its incident vertices), and so obtain a new graph $\mathcal{G}'$. There is a bijection between the oriented edges of the graph $\mathcal{G} \setminus \{e\}$ and those of the graph $\mathcal{G}'$.

Now we show that the total number of non-backtracking paths from i to j of length less than $\text{dist}_{\mathcal{G}}(i, j) + k = \text{dist}_{\mathcal{G}'}(i, j) + k$ in $\mathcal{G}'$ is at least t_k. Let $(\vec{e}_1, \vec{e}_2, \vec{e}_3, \dots)$ be any non-backtracking path from i to j in the graph $\mathcal{G}$ that is not a geodesic. If some $\vec{e}_\beta$ is (x, y) or (y, x), we remove it from the path and view the remaining part as a path from i to j in the graph $\mathcal{G}'$. In this way we get a shorter path from i to j in $\mathcal{G}'$. The new path is still non-backtracking, and we can recover the original path in $\mathcal{G}$ from the new path in $\mathcal{G}'$ since $x \neq y$. Therefore the total number of non-backtracking paths from i to j of length less than $\text{dist}_{\mathcal{G}}(i, j) + k = \text{dist}_{\mathcal{G}'}(i, j) + k$ in $\mathcal{G}'$ is at least t_k.

We repeat this procedure with edges e (not on a geodesic) chosen arbitrarily as long as possible. This creates a new graph $\mathcal{G}_1$ (which may depend on the choice of edges in the steps) with vertex set $\mathbb{G}_1$. By construction, the edges in $\mathcal{G}_1$ are either self-loops or on geodesics from i to j. Thus the vertex set of $\mathcal{G}_1$ decomposes into

$$(A.5) \quad \mathbb{G}_1 = \mathbb{V}_0 \cup \mathbb{V}_1 \cup \cdots \cup \mathbb{V}_{\text{dist}_{\mathcal{G}_1}(i,j)}, \quad \text{where } \mathbb{V}_m := \{v \in \mathbb{G}_1 : \text{dist}_{\mathcal{G}_1}(i, v) = m\},$$

or equivalently, $\mathbb{V}_{\text{dist}_{\mathcal{G}_1}(i,j)-m} := \{v \in \mathbb{G}_1 : \text{dist}_{\mathcal{G}_1}(v, j) = m\}$. In particular, $\mathbb{V}_0 = \{i\}$ and $\mathbb{V}_{\text{dist}_{\mathcal{G}_1}(i,j)} = \{j\}$. Any edge in $\mathcal{G}_1$ is either a self-loop or has one vertex in $\mathbb{V}_m$ and the other vertex in $\mathbb{V}_{m+1}$, for some $m \in [\![0, \text{dist}_{\mathcal{G}_1}(i, j) - 1]\!]$. The excess of $\mathcal{G}_1$ is ω.

Step 2. Given two edges $e = \{v_m, v_{m+1}\}$ and $e' = \{v_m, v'_{m+1}\}$ with $v_m \in \mathbb{V}_m$ and $v_{m+1} \neq v'_{m+1} \in \mathbb{V}_{m+1}$, we remove the edge e' and identify v'_{m+1} with v_{m+1}, thus creating a new graph $\mathcal{G}'_1$. Again there is a bijection between the oriented edges of the graph $\mathcal{G}_1 \setminus \{e'\}$ and those of the graph $\mathcal{G}'_1$.

Now we show that the total number of non-backtracking paths from i to j of length less than $\mathrm{dist}_{\mathcal{G}}(i,j)+k = \mathrm{dist}_{\mathcal{G}'_1}(i,j)+k$ in $\mathcal{G}'_1$ is at least t_k. Let $(\vec{e}_1, \vec{e}_2, \vec{e}_3, \ldots)$ be any non-backtracking path from i to j in the graph $\mathcal{G}_1$. If $\vec{e}_\beta = (v'_{m+1}, v_m)$ and $\vec{e}_{\beta+1} \neq (v_m, v_{m+1})$, we replace $\vec{e}_\beta$ by (v_{m+1}, v_m); if $\vec{e}_\beta = (v'_{m+1}, v_m)$ and $\vec{e}_{\beta+1} = (v_m, v_{m+1})$, we remove both $\vec{e}_\beta$ and $\vec{e}_{\beta+1}$; if $\vec{e}_\beta = (v_m, v'_{m+1})$ and $\vec{e}_{\beta-1} \neq (v_{m+1}, v_m)$, we replace $\vec{e}_\beta$ by (v_m, v_{m+1}); if $\vec{e}_\beta = (v_m, v'_{m+1})$ and $\vec{e}_{\beta-1} = (v_{m+1}, v_m)$, we remove both $\vec{e}_\beta$ and $\vec{e}_{\beta-1}$. Then we view the remaining part as a path from i to j in the graph $\mathcal{G}'_1$, whose length is at most as long as that of the original path. The new path is still non-backtracking, we can recover the original path in $\mathcal{G}_1$ from the new path in $\mathcal{G}'_1$ since $v_{m+1} \neq v'_{m+1}$. Therefore the total number of non-backtracking paths from i to j of length less than $\mathrm{dist}_{\mathcal{G}}(i,j)+k = \mathrm{dist}_{\mathcal{G}'_1}(i,j)+k$ in $\mathcal{G}'_1$ is at least t_k.

For any $m \in [\![0, \mathrm{dist}_{\mathcal{G}_1}(i,j) - 2]\!]$, if in the new graph $|\{v : \mathrm{dist}_{\mathcal{G}_1}(i,v) = m+1\}| \geqslant 2$, we can repeat the above process to reduce it by one. We repeat this procedure as long as possible, choosing at every step edges e and e' arbitrarily such that the conditions are satisfied. Finally, we obtain a graph $\mathcal{G}_2$ (which again is not unique) that has exactly $\mathrm{dist}_{\mathcal{G}_2}(i,j) + 1$ vertices, $\{v_0 = i, v_1, v_2, \ldots, v_{\mathrm{dist}_{\mathcal{G}_2}(i,j)} = j\}$, such that $\mathrm{dist}_{\mathcal{G}_2}(i, v_m) = m$ for $m \in [\![0, \mathrm{dist}_{\mathcal{G}_2(i,j)}]\!]$. The excess of $\mathcal{G}_2$ is ω.

Step 3. In the final step, given any edge e from v_m to v_{m+1}, if it is the only edge from v_m to v_{m+1}, we shrink it to a point. This preserves non-backtracking paths, and it reduces the distance between i and j by one. By shrinking all edges of multiplicity one, we obtain a graph $\mathcal{G}_3$. The number of non-backtracking paths from i to j of length less than $\mathrm{dist}_{\mathcal{G}_3}(i,j) + k$ is at least t_k, and the excess of $\mathcal{G}_3$ is ω.

Final step. To bound the number of non-backtracking paths from i to j in $\mathcal{G}$, it suffices to estimate the number of non-backtracking paths from i to j in the graph $\mathcal{G}_3$. Let $\ell = \mathrm{dist}_{\mathcal{G}_3}(i,j)$, s be the total number of self-loops in $\mathcal{G}_3$, $w_m + 1$ the multiplicity of the edge $\{v_{m-1}, v_m\}$, for $m \in [\![1, \ell]\!]$, and set $w = \max_{1 \leqslant m \leqslant \ell} w_m$. Since $\mathcal{G}_3$ has excess ω, $s + \sum_{m=1}^{\ell} w_m = \omega$. The maximum degree of the graph $\mathcal{G}_3$ is bounded by

$2s + 2w + 2$. Now any non-backtracking path from i to j of length $\ell + k$ necessarily contains the edges $(v_0, v_1), (v_1, v_2), \ldots, (v_{\ell-1}, v_\ell)$, and for each of them there are $w_1 + 1, w_2 + 2, \ldots, w_\ell + 1$ choices respectively. For other steps, there are at most $2s + 1 + 2w$ choices. The total number of such paths is bounded by

$$(\text{A.6}) \qquad \binom{\ell + k}{\ell} (2s + 1 + 2w)^k \prod_{m=1}^{\ell} (w_m + 1),$$

under the condition $s + \sum_{m=1}^{\ell} w_m = \omega$. Note that (A.6) increases if we decrease s by 1 and increase some w_m in such a way that w increases by 1. Therefore (A.6) achieves its maximum at $s = 0$. We denote

$$a_k := \binom{\ell + k}{\ell} (1 + 2w)^k \prod_{m=1}^{\ell} (w_m + 1),$$

Since $1 + n \leqslant 2^n$ for any $n \in \mathbb{N}_0$ and $\sum_{m=1}^{\ell} w_m = \omega$, we then have $a_0 \leqslant \prod_{m=1}^{\ell} (w_m + 1) \leqslant 2^\omega$. For a_k with $k \geqslant 1$, notice that $\omega = \sum_{m=1}^{\ell} w_m \geqslant w + (\ell - 1)$ so that $w \leqslant \omega - (\ell - 1)$, and thus

$$a_k \leqslant \frac{\ell + k}{k}(1 + 2w)a_{k-1} \leqslant (\ell + 1)(2\omega - 2\ell + 3)a_{k-1} \leqslant \frac{(2\omega + 5)^2}{8}a_{k-1} \leqslant (2^\omega - 1)a_{k-1},$$

given that $\omega \geqslant 6$. Therefore

$$t_k \leqslant a_0 + a_1 + \cdots + a_{k-1} \leqslant 2^\omega \left(1 + (2^\omega - 1) + \cdots (2^\omega - 1)^{k-1}\right) \leqslant 2^{\omega k}.$$

This finishes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Proof of (2.9). Let $\mathbb{H}$ be the vertex set of $\mathcal{H}$, ω_0 be the excess of the subgraph $\mathcal{H}$, and $\bar{\mathcal{H}}$ the subgraph induced by $\mathcal{G}$ on $\mathbb{H}$. If $\text{dist}_\mathcal{G}(i, j) \geqslant \ell + 1$, then (2.8) implies

$\#\{\text{non-backtracking paths from } i \text{ to } j \text{ of length } \ell + k, \text{ not completely in } \mathcal{H}\}$

$\leqslant \#\{\text{non-backtracking paths from } i \text{ to } j \text{ of length } \ell + k\} \leqslant 2^{\omega k},$

and the claim (2.9) follows. Therefore, in the following, assume that $\text{dist}_\mathcal{G}(i,j) \leqslant \ell$, and also that $\mathcal{H}, \mathcal{G}$ are connected (otherwise, we can replace $\mathcal{H}$ by its connected component containing i and j, and $\mathcal{G}$ by its connected component containing $\mathcal{H}$). For any non-backtracking path from i to j which is not completely contained in $\mathcal{H}$, let $\vec{e}$ be the first edge in the path which does not belong to $\mathcal{H}$. There are three possibilities for such edge e: (i) $e \in \bar{\mathcal{H}}$. We denote the set of such edges by E_1. (ii) If we remove e from $\mathcal{G}$, then $\mathcal{G} \setminus \{e\}$ breaks into two connected components. It is necessary that the component not containing i, j contains cycles. We denote the set of such edges by E_2. (iii) $e \notin \bar{\mathcal{H}}$, and if we remove e from $\mathcal{G}$, $\mathcal{G} \setminus \{e\}$ is still connected. We denote the set of such edges by E_3.

We consider the graph $\mathcal{G} \setminus \{E_1 \cup E_2 \cup E_3\}$, from $\mathcal{G}$ by removing edges $E_1 \cup E_2 \cup E_3$. It consists some many connected components, one corresponds to the graph $\mathcal{H}$, others are in one-to-one correspondence with the connected components of $\mathcal{G} \setminus \bar{\mathcal{H}}$, the graph from removing $\mathcal{H}$ from $\mathcal{G}$. Notice from the definition of these edge sets E_1, E_2, E_3, each connected component of $\mathcal{G} \setminus \bar{\mathcal{H}}$ contains exactly one edges in E_2 or at least two edges in E_3. Therefore, $\mathcal{G} \setminus \{E_1 \cup E_2 \cup E_3\}$ has at most $1 + |E_2| + |E_3|/2$ connected components, where 1 represents the component $\mathcal{H}$. For the excess of $\mathcal{G} \setminus \{E_1 \cup E_2 \cup E_3\}$, since its subgraph $\mathcal{H}$ has excess ω_0, and each new components, due to removing of edges in E_2, has excess at least 1, $\mathcal{G} \setminus \{E_1 \cup E_2 \cup E_3\}$ has excess at least $\omega_0 + |E_2|$.

Claim A.1.

(A.7)
$$2|E_1| + |E_2| + |E_3| \leqslant 2(\omega - \omega_0)$$

Proof. To prove (A.7), for any finite graph $\mathcal{X}$, set

(A.8)
$$\chi(\mathcal{X}) = \#\text{connected components}(\mathcal{G}_0) - \text{excess}(\mathcal{G}_0).$$

By the definition of excess, $\chi(\mathcal{X}) = \#\text{vertices}(\mathcal{G}_0) - \#\text{edges}(\mathcal{G}_0)$, we have $\chi(\mathcal{X} \setminus e) = \chi(\mathcal{X}) + 1$ for any graph $\mathcal{X}$ and any edge e in $\mathcal{X}$. Since the graph $\mathcal{G}$ is connected and

has excess at most ω, it follows that $\chi(\mathcal{G}) \geqslant 1 - \omega$. Thus if we remove $E_1 \cup E_2 \cup E_3$ from $\mathcal{G}$, the remaining graph has excess at least $\omega_0 + |E_2|$ and at most $1 + |E_2| + |E_3|/2$ connected components. Therefore

$$1 + |E_2| + |E_3|/2 - |E_2| - \omega_0 \geqslant \chi(\mathcal{G} \setminus (E_1 \cup E_2 \cup E_3)) \geqslant 1 - \omega + |E_1| + |E_2| + |E_3|,$$

and thus $|E_1| + |E_2| + |E_3|/2 \leqslant \omega - \omega_0$. (A.7) follows. $\qquad\square$

In the following we count the number of length $\ell + k$ non-backtracking paths from i to j, containing $\vec{e} = (i_1, j_1)$ as the first edge not in $\mathcal{H}$, i.e., $\{i_1, j_1\} \in E_1 \cup E_2 \cup E_3$. Let $\mathrm{dist}_{\mathcal{G}}(i, i_1) = \ell_1$ and $\mathrm{dist}_{\mathcal{G}}(j_1, j) = \ell_2$. Since $\{i_1, j_1\}$ is not in $\mathcal{H}$, it is necessary that $\ell_1 + \ell_2 \geqslant \ell$. Thus, $\vec{e}$ must be the $\ell_1 + 1, \ell_1 + 2, \ldots$, or $(\ell_1 + k)$-th step in the path. The total number of such non-backtracking paths is bounded by

$$\sum_{k_1=1}^{k} \#\{\text{non-backtracking paths from } i \text{ to } i_1 \text{ of length } \ell_1 + k_1 - 1, \text{ in } \mathcal{H}\}$$

$$\times \#\{\text{non-backtracking paths from } j_1 \text{ to } j \text{ of length } \ell + k - \ell_1 - k_1, \text{ in } \mathcal{G}\}$$

$$\leqslant \sum_{k_1=1}^{k} 2^{\omega_0 k_1} 2^{\omega(k - k_1 + 1)} \leqslant 2^{\omega(k+1)} \sum_{k_1=1}^{k} 2^{(\omega_0 - \omega)k_1}.$$

Since by (A.7), $2|E_1| + |E_2| + |E_3| \leqslant 2(\omega - \omega_0)$, there are at most $2(\omega - \omega_0)$ choices for the oriented edge $\vec{e}$, the total number of such non-backtracking paths is bounded by

$$\#\{\text{non-backtracking paths from } i \text{ to } j \text{ of length } \ell + k, \text{ not completely in } \mathcal{H}\}$$

$$\leqslant 2(\omega - \omega_0) 2^{\omega(k+1)} \sum_{k_1=1}^{k} 2^{(\omega_0 - \omega)k_1} \leqslant 2^{\omega(k+1)+1}.$$

This completes the proof. $\qquad\square$

A.3. **Proof of Lemma 3.35.** To understand the distances $\mathrm{dist}_{\tilde{\mathcal{G}}}(x, i)$ for all $i \in \mathbb{T}_\ell$, we need some more notations. A *simple pruning*[46, Definition 4.4] is the operation

of removing one leaf and its incident edge from a graph. By repeating pruning on the graph $\tilde{\mathcal{G}}_0$, we get a graph $\tilde{\mathcal{G}}_2$ with vertex set $\tilde{\mathbb{G}}_2$, such that it contains at most two leave vertices: 1 and x.

Claim A.2.

(A.9) $$|\tilde{\mathbb{G}}_2 \cap \mathbb{T}_k| \leqslant 2\omega + 1, \quad 0 \leqslant k \leqslant \ell.$$

Proof. For $k = 0$, (A.9) holds trivially, $|\tilde{\mathbb{G}}_2 \cap \mathbb{T}_0| = 1 \leqslant 2\omega + 1$. For $k \geqslant 1$, say $\tilde{\mathbb{G}}_2 \cap \mathbb{T}_k = \{v_1, v_2, \dots, v_m\}$. By our construction of $\tilde{\mathcal{G}}_2$, there are vertices $v_1', v_2', \dots, v_m' \in \mathbb{T}_{k-1}$ such that the edges $\{v_1', v_1\}, \{v_2', v_2\}, \dots, \{v_m', v_m\} \in \tilde{\mathcal{G}}_2$. For any $i \in [\![1, m]\!]$, if we remove the edge $\{v_i', v_i\}$ from $\tilde{\mathcal{G}}_2$, the graph $\tilde{\mathcal{G}}_2$ will either still be connected; or it will break into two connected components, one contains vertex 1, and the other contains vertex x or some cycles. Let m_1, m_2 the number of edges in the first case and second case respectively. If we remove all edges $\{v_1', v_1\}, \{v_2', v_2\}, \dots, \{v_m', v_m\}$, there will be at most $1 + m_1/2 + m_2$ connected components, and at least excess $m_2 - \mathbf{1}_{m_2 > 0}$ left. Notice that the graph $\tilde{\mathcal{G}}_2$ is connected and has excess at most ω. Recall the function χ as in (A.8), we have

$$1 + m_2 + m_1/2 - (m_2 - \mathbf{1}_{m_2 > 0}) \geqslant \chi(\tilde{\mathcal{G}}_2 \setminus \{\{v_1', v_1\}, \dots, \{v_m', v_m\}\}) \geqslant 1 - \omega + m_1 + m_2.$$

Therefore $m_1 + m_2 \leqslant 2\omega + 1$, and the claim follows. $\qquad \square$

With the above preparations, we can prove Lemma 3.35 as follows.

Proof of Lemma 3.35. Fix a geodesic $\mathcal{P}$ (viewed as a sequence of oriented edges) in $\tilde{\mathcal{G}}_0$ from vertex x to vertex $i \in \mathbb{T}_\ell$, there are three possibilities for its step (v', v): (i) the edge is downward, i.e. $\operatorname{dist}_{\tilde{\mathcal{G}}_0}(1, v) = \operatorname{dist}_{\tilde{\mathcal{G}}_0}(1, v') + 1$; (ii) the edge is horizontal, i.e. $\operatorname{dist}_{\tilde{\mathcal{G}}_0}(1, v) = \operatorname{dist}_{\tilde{\mathcal{G}}_0}(1, v')$, in this case $v \in \tilde{\mathbb{G}}_2$; (iii) the edge is upward, i.e. $\operatorname{dist}_{\tilde{\mathcal{G}}_0}(1, v) = \operatorname{dist}_{\tilde{\mathcal{G}}_0}(1, v') - 1$, in this case $v \in \tilde{\mathbb{G}}_2$. We denote (v', v) the last step in $\mathcal{P}$, which is horizontal or upward. Then $v \in \tilde{\mathbb{G}}_2$ and we say the vertex i is associated

with the vertex v (which may not be unique). By our choice of (v', v), the steps from v to i in $\mathcal{P}$ are all downward, thus $v \in \mathbb{T}$. Moreover we have the estimate: for any vertex $i \in \mathbb{T}_\ell$ associated with v

$$\text{dist}_{\tilde{\mathcal{G}}_0}(x, i) \geqslant \left|\text{dist}_{\tilde{\mathcal{G}}_0}(1, x) - \text{dist}_{\tilde{\mathcal{G}}_0}(1, v)\right| + \text{dist}_{\tilde{\mathcal{G}}_0}(v, i)$$

$$= \left|\text{dist}_{\tilde{\mathcal{G}}_0}(1, x) - \text{dist}_{\tilde{\mathcal{G}}_0}(1, v)\right| + \left|\ell - \text{dist}_{\tilde{\mathcal{G}}_0}(1, v)\right|.$$

Especially, if $v \in \mathbb{T}_{\ell_3}$, i.e. v is distance ℓ_3 from vertex 1, the above relation simplifies to

$$\text{dist}_{\tilde{\mathcal{G}}_0}(x, i) \geqslant |\ell_1 - \ell_3| + (\ell - \ell_3),$$

and by noticing $q < 1$, we have

(A.10)
$$q^{\text{dist}_{\tilde{\mathcal{G}}_0}(x,i)} \leqslant \begin{cases} q^{\ell + \ell_1 - 2\ell_3}, & \text{if } \ell_3 \leqslant \ell_1, \\ q^{\ell - \ell_1}, & \text{if } \ell_3 \geqslant \ell_1. \end{cases}$$

In this way, each vertex $i \in \mathbb{T}_\ell$ is associated with some vertex $v \in \tilde{\mathbb{G}}_2$. If $v \in \tilde{\mathbb{G}}_2 \cap \mathbb{T}_{\ell_3}$, the total number of vertices in $\mathbb{T}_\ell$ associated with v is at most $(d-1)^{\ell-\ell_3}$, since they are all distance $\ell - \ell_3$ away from v. The total number of vertices $i \in \mathbb{T}_\ell$ associated with some $v \in \tilde{\mathbb{G}}_2 \cap \{\mathbb{T}_{\ell_1} \cup \mathbb{T}_{\ell_1+1} \cdots \cup \mathbb{T}_\ell\}$ is bounded by $(2\omega+1)(1+(d-1)+\cdots+(d-1)^{\ell_1}) \leqslant 2(\omega+1)(d-1)^{\ell_1}$, provided that $d \geqslant 2\omega + 3$. Notice that we have the decomposition

$$\{q^{\text{dist}_{\tilde{\mathcal{G}}_0}(x,i)} : i \in \mathbb{T}_\ell\} = \cup_{\ell_3 \in [\![0, \ell_1-1]\!]}\{q^{\text{dist}_{\tilde{\mathcal{G}}_0}(x,i)} : i \in \mathbb{T}_\ell, i \text{ is associated with some } v \in \tilde{\mathbb{G}}_2 \cap \mathbb{T}_{\ell_3}\}$$

$$\cup \{q^{\text{dist}_{\tilde{\mathcal{G}}_0}(x,i)} : i \in \mathbb{T}_\ell, i \text{ is associated with some } v \in \tilde{\mathbb{G}}_2 \cap \{\mathbb{T}_{\ell_1} \cup \mathbb{T}_{\ell_1+1} \cdots \cup \mathbb{T}_\ell\}\}.$$

Lemma 3.35 follows by combining with (A.10) and (A.9). $\qquad\square$

APPENDIX B. PROPERTIES OF THE GREEN'S FUNCTIONS

Throughout this paper, we repeatedly use some (well-known) identities for Green's functions, which we collect in this appendix.

B.1. **Resolvent identity.** The following well-known identity is referred as resolvent identity: for two invertible matrices A and B of the same size, we have

$$\text{(B.1)} \qquad A^{-1} - B^{-1} = A^{-1}(B - A)B^{-1} = B^{-1}(B - A)A^{-1}.$$

B.2. **Schur complement formula.** Given an $N \times N$ matrix M and an index set $\mathbb{T} \subset [\![N]\!]$, recall that we denote by $M|_{\mathbb{T}}$ the $\mathbb{T} \times \mathbb{T}$-matrix obtained by restricting M to $\mathbb{T}$, and that by $M^{(\mathbb{T})} = M|_{[\![N]\!] \setminus \mathbb{T}}$ we denote the matrix obtained by removing the rows and columns with indices in $\mathbb{T}$. Thus, for any $\mathbb{T} \subset [\![N]\!]$, any symmetric matrix H can be written (up to rearrangement of indices) in the block form

$$\text{(B.2)} \qquad H = \begin{bmatrix} A & B' \\ B & D \end{bmatrix},$$

with $A = H|_{\mathbb{T}}$ and $D = H^{(\mathbb{T})}$. The Schur complement formula asserts that, for any $z \in \mathbb{C}_+$,

(B.3)
$$G = (H - z)^{-1} = \begin{bmatrix} (A - B'G^{(\mathbb{T})}B)^{-1} & -(A - B'G^{(\mathbb{T})}B)^{-1}B'G^{(\mathbb{T})} \\ -G^{(\mathbb{T})}B(A - B'G^{(\mathbb{T})}B)^{-1} & G^{(\mathbb{T})} + G^{(\mathbb{T})}B(A - B'G^{(\mathbb{T})}B)^{-1}B'G^{(\mathbb{T})} \end{bmatrix},$$

where $G^{(\mathbb{T})} = (D - z)^{-1}$. Throughout the paper, we often use the following special cases of (B.3):

$$G|_{\mathbb{T}} = (A - B'G^{(\mathbb{T})}B)^{-1},$$

$$\text{(B.4)} \qquad G|_{\mathbb{T}^c} - G^{(\mathbb{T})} = G|_{\mathbb{T}^c\mathbb{T}}(G|_{\mathbb{T}})^{-1}G|_{\mathbb{T}\mathbb{T}^c},$$

$$G|_{\mathbb{T}\mathbb{T}^c} = -G|_{\mathbb{T}}B'G^{(\mathbb{T})},$$

as well as the special case

$$\text{(B.5)} \qquad G_{ij}^{(k)} = G_{ij} - \frac{G_{ik}G_{kj}}{G_{kk}}.$$

B.3. **Ward identity.** For any symmetric $N \times N$ matrix H, its Green's function $G(z) = (H - z)^{-1}$ satisfies the *Ward identity*

$$\text{(B.6)} \qquad \sum_{j=1}^{N} |G_{ij}(z)|^2 = \frac{\text{Im}\, G_{jj}(z)}{\eta},$$

where $\eta = \text{Im}[z]$. This identity follows from (B.1) with $A = H - z$ and $B = (H - z)^*$. In particular, (B.6) provides a bound for the sum $\sum_{j=1}^{N} |G_{ij}(z)|^2$ in terms of the diagonal of the Green's function. For an explanation why this algebraic identity has the interpretation of a Ward, see e.g. [83, p.147].

B.4. **Covering map.** For any vertex i, the vector $(G_{i1}, G_{i2}, G_{i3}, \dots) \in \ell^2(\llbracket N \rrbracket)$ is uniquely determined by the following relations:

$$\text{(B.7)} \qquad \begin{aligned} 1 + zG_{ii} &= \frac{1}{\sqrt{d - 1}} \sum_{k: i \sim k} G_{ik} \\ zG_{ij} &= \frac{1}{\sqrt{d - 1}} \sum_{k: j \sim k} G_{ik} \end{aligned}$$

where $l \sim k$ denotes that l and k are adjacent in $\mathcal{G}$, i.e., that $A_{kl} = 1$.

Lemma B.1. *Given a covering* $\pi : \tilde{\mathcal{G}} \to \mathcal{G}$ *of graphs, denote the Green's function of* $\tilde{\mathcal{G}}$ *by* $\tilde{G}$ *and that of* $\mathcal{G}$ *by* G. *Then for all vertices* i, j *in* $\mathcal{G}$, *the Green's functions obey*

$$\text{(B.8)} \qquad G_{ij} = \sum_{y: \pi(y) = j} \tilde{G}_{xy}.$$

Proof of (B.8). We give the proof for simple graphs $\mathcal{G}, \tilde{\mathcal{G}}$. (The statement also holds for graphs with self-loops and multiple edges if $\sum_{k: i \sim k}$ is interpreted as the sum of all the oriented edges (i, k); especially, a self-loop should be counted twice.) Clearly, $\tilde{G}$ satisfies the relations (B.7) with G replaced by $\tilde{G}$. For any fixed $x \in \tilde{\mathcal{G}}$ such that $\pi(x) = i$, we can define:

$$\text{(B.9)} \qquad G_{ij} = \sum_{y: \pi(y) = j} \tilde{G}_{xy},$$

if the right-hand side is summable. Assuming that for any j the right-hand side of (B.9) is well defined, we verify that $(G_{ij})_j$ satisfies the relation (B.7), and thus that it gives the Green's function of H. Indeed,

$$1 + zG_{ii} = 1 + z \sum_{y:\pi(y)=i} \tilde{G}_{xy} = 1 + z\tilde{G}_{xx} + z \sum_{y:\pi(y)=i, y\neq x} \tilde{G}_{xy}$$

$$= \frac{1}{\sqrt{d-1}} \sum_{w:w\sim x} \tilde{G}_{xw} + \frac{1}{\sqrt{d-1}} \sum_{y:\pi(y)=i, y\neq x} \sum_{w:w\sim y} \tilde{G}_{xw} = \frac{1}{\sqrt{d-1}} \sum_{y:\pi(y)=i} \sum_{w:w\sim y} \tilde{G}_{xw}.$$

Since there is no self-loop and multi-edge in our graph $\mathcal{G}$, for any $y_1 \neq y_2$ with $\pi(y_1) = \pi(y_2) = i$ and $w_1 \sim y_1$ and $w_2 \sim y_2$, it is necessary that $w_1 \neq w_2$. Therefore:

$$\frac{1}{\sqrt{d-1}} \sum_{y:\pi(y)=i} \sum_{w:w\sim y} \tilde{G}_{xw} = \frac{1}{\sqrt{d-1}} \sum_{k:i\sim k} \sum_{w:\pi(w)=k} \tilde{G}_{xw} = \frac{1}{\sqrt{d-1}} \sum_{k:i\sim k} G_{ik}.$$

Similarly, for the second relation (B.7),

$$zG_{ij} = z \sum_{y:\pi(y)=j} \tilde{G}_{xy} = \frac{1}{\sqrt{d-1}} \sum_{y:\pi(y)=j} \sum_{w:w\sim y} \tilde{G}_{xw} = \frac{1}{\sqrt{d-1}} \sum_{k:j\sim k} \sum_{w:\pi(w)=k} \tilde{G}_{xw} = \frac{1}{\sqrt{d-1}} \sum_{k:j\sim k} G_{ik},$$

as needed. $\qquad\square$

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